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Ideals in group rings of soluble groups of finite rank. (English) Zbl 0532.16007

Math. Proc. Camb. Philos. Soc. 97, 27-49 (1985).

The bulk of this paper is a study of Γ-ideals of the group algebra kA where Γ is a group acting on a torsion-free abelian group A of finite rank. Define the standardiser of an ideal I of kA to be the subgroup of Γ of elements γ such that $I \cap kA_1 = I^\gamma \cap kA_1$ for some finitely generated (f.g.) subgroup A_1 (dependent on γ) of rank $r(A)$. Every Γ-ideal has at least one minimal prime over it almost standardised by Γ. This is significant because faithful almost standardised primes of kA are controlled by $\Delta_\Gamma(A)$, the subgroup of Γ-orbital elements of A (Theorem A). The proof is based upon that of Theorem D of *J. E. Roseblade* [Proc. Lond. Math. Soc. (3) 36, 385–447 (1978; Zbl 0391.16008)], the analogous result for f.g. A. This was a generalisation of a theorem of *G. M. Bergman* [Trans. Am. Math. Soc. 157, 459–470 (1971; Zbl 0197.17102)] which itself may be extended to the finite rank case. A is said to be a plinth if it is rationally irreducible as a $\mathbb{Z}\Gamma_1$-module for all Γ_1 of finite index in Γ. Part of Theorem C states: Let A be a plinth which has the maximum condition on Γ-invariant subgroups. Then kA has the maximum condition on Γ-ideals and a non-zero Γ-ideal I satisfies $\dim_k(kA/I) < \infty$. Furthermore if Γ is f.g. nilpotent and G is any group containing A as a normal subgroup with $G/A \cong \Gamma$ then kG has the maximum condition on ideals. This is significant because it provides numerous examples on non-Noetherian rings with the maximum condition on ideals. Moreover there are groups G with kG, but not $\mathbb{Z}G$, having that property.

On the other hand the results about Γ-ideals do not carry over from the f.g. to the finite rank case. For fields of characteristic zero faithful primes need not be controlled by $\Delta_\Gamma(A)$ even when A is the direct product of two plinths. For f.g. abelian Γ a condition is obtained in terms of the valuation sphere introduced by *R. Bieri* and *R. Strebel* [Proc. Lond. Math. Soc. (3) 41, 439–464 (1980; Zbl 0448.20029)], sufficient for the existence of this departure from polycyclic behaviour. This condition also proves to be a necessary one when Γ is infinite cyclic. This effectively provides a characterisation of the actions of an infinite cyclic group Γ for which all faithful Γ-primes are annihilator-free.

Two applications of this study of Γ-ideals are:

Corollary E. Let G be a soluble group of finite rank with no non-trivial finite normal subgroups and k be a field. Then all annihilator-free prime ideals of kG are controlled by $\Delta(G)$.

Theorem F. Let G be a soluble group of finite rank and k be a non-absolute field. Then kG is primitive if $\Delta(G) = 1$.

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MSC:

16S34	Group rings
20C07	Group rings of infinite groups and their modules (group-theoretic aspects)
16D25	Ideals in associative algebras
20F16	Solvable groups, supersolvable groups
16D60	Simple and semisimple modules, primitive rings and ideals in associative algebras

Keywords:

primitive group rings; gamma-ideals of group algebras; standardisers of ideals; minimal primes; faithful almost standardised primes; orbital elements; maximum condition on ideals; plinths; valuation sphere; soluble groups of finite rank; annihilator-free prime ideals

Full Text: DOI

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