Hörmander, Lars

The analysis of linear partial differential operators. I: Distribution theory and Fourier analysis. (English) Zbl 0521.35001

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The first volume of L. Hörmander's treatise is entirely devoted to distribution theory and the corresponding harmonic analysis. It certainly will be a classic for many years to come, and this for several reasons, which will become apparent in the sequel. This volume has 9 chapters. Each chapter ends with notes, giving references and motivations. Various and illuminating examples are scattered all along and they are very useful, especially when reading the second volume.

The first, "Test functions", begins with a brief review of differential calculus, followed by a paragraph on the existence of test functions, and another about convolution. Here one finds some refined results about regularization (th. 1.3.5), the Denjoy-Carleman theorem about the characterisation of quasi-analyticity, etc. The chapter ends with a paragraph about cut-off functions and partitions of unity where again one finds some refined results (due mainly to Whitney) which will be of use later on.

Chapter II "Definition and basic properties of distributions" begins with a careful study of the continuity on $C_0(X)$, X an open subset of $\mathbb{R}^n$ (three equivalent definitions are given, (th. 2.14, th. 2.15), without using explicitly the theory of inductive limits of locally convex spaces).

The second paragraph, "Localization" introduces the support of a distribution and shows (without actually using the word) that $X \to \mathcal{D}'(X)$ is in fact a sheaf. The next paragraph "Distributions with compact supports", besides the usual things expected to be found there, contains a useful splitting theorem (th. 2.3.5), the celebrated Whitney extension theorem and, as applications, some useful estimates for distributions with contact support (cor, 2.3.8 and th. 2.3.11).

Chapter III, "Differentiation and multiplication by functions" begins with the definition of the derivative of a distribution; then it shows how to calculate the derivative of a function with jumps (th. 3.13) etc. Among the results we mention the very useful th. 3.19 which gives the results of differentiation of u_{X_Y} where X_Y is the characteristic function of an open set Y with C^1 boundary, and th. 3.1.10. Very useful for the sequel is also the study of the existence of boundary values of analytic functions in the sense of distribution theory (th. 3.1.11, th. 3.1.12, and th. 3.1.15). The next three sections of a more computational nature contain a wealth of interesting examples of distributions as for instance homogeneous distributions, some of them fundamental solutions of classical differential operators (Laplace operator, Schrödinger operator, the heat operator, Cauchy-Riemann operator).

Chapter IV "Convolution" contains also many very interesting and useful results. We mention th. 4.17 and the very nice th. 4.18 concerning the characterization of subharmonic distributions. Then the plurisubharmonic functions are introduced (in fact and as a non trivial result, the Lelong result about the current defined by an analytic set is proved (th. 4.1.12 and th. 4.1.13) culminating with th. 4.1.15 about Lelong's number (in fact the word current is not used at all, but that is not important). Paragraph 4.2 introduces the convolution of distributions, using the fact that if a linear map U from $c_0^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$ is sequentially continuous and commutes with translations, then it must be of the form $U(\varphi) = u * \varphi$ for $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$. Thus the author can avoid using tensor products, but conceptually I don't think that the present approach (although perfectly correct) is the best choice. Of course, it was a deliberate choice, as explained by the author, paragraph 4.3 is devoted to the proof of the theorem of supports, and 4.4 dedicated to the proof of the theorem of support, and 4.4 clarifies the role of fundamental solutions (we note an extension of Runge's approximation theorem, th. 4.4.5), the existence th. 4.4.6 and the representation theorem 4.4.7, which yields several important consequences). Paragraph 4.5 "Basic L^p estimates" culminates with the Sobolev embedding theorem (th. 4.5.13), whose proof is derived in Zygmund's spirit, using the Zygmund-Calderon covering lemma and Marcinkiewicz's argument to give first the Hardy-Littlewood-Sobolev inequality (th. 4.5.3).

The short chapter V, "Distribution in product spaces", is mainly dedicated to the proof of Schwartz's kernel theorem. The proof uses the fact that a regularization of test functions in the product space by a product of two test functions in each space can be considered as a superposition of tensor products of test functions, and thus one has first to define convolution and then the tensor product; this explains the somehow artificial way to introduce convolution in chapter IV.

Chapter VI, "Composition with smooth map" has a more geometrical content. First one defines the pull-back of a smooth map, then in §6.2 using the properties of the pullback one considers some fundamental solutions of second-order differential operators; in particular, a careful study of the wave operators is made, and we are thus quickly led to the solution of the Cauchy problem for the wave equation (th. 6.24). Next, in §6.3 the distributions on a manifold are introduced, and in §6.4 "The tangent and cotangent bundles", after starting with the definitions of a vector bundle, of the tangent and cotangent bundles and after giving some facts and some relevant definitions (of the conormal one form on the cotangent bundle, the conormal bundle, the principal symbol p of a differential operator P, the characteristics of a differential operator, the symplectic form σ, the Hamiltonian vector field H_p of p, the bicharacteristic curve of p) the author gives a very careful study of the Hamilton-Jacobi theory of integration of first-order partial differential equations, both to geometric and in analytic form.

Chapter VII "The Fourier transformation" has over 90 pages and is something very impressive and now most of the results accumulated to preceding chapters begin to show their usefulness. The first section, although- starting from the definition, contains some classical results as the Riesz-Thorin interpolation theorem, Hausdorff-Young inequality, some interesting results on homogeneous distributions and their Fourier transforms, some applications to operators with constant coefficients. Let us mention some non-trivial calculations concerning the Fourier transforms of some distributions (th. 7.1.23 or th. 7.1.24). Remark also the results concerning Fourier transforms of simple layers (densities) on submanifolds (th. 7.1.25, th. 7.1.26, 7.1.28). Section 7.2 contains the proof of Poisson's summation formula and some facts about periodic distributions.

The Fourier-Laplace transformation in $\mathcal{E}'$ is studied in section 7.3; it begins with the Paley-Wiener-Schwartz theorem, then Malgrange's theorem (th. 7.3.2) about the existence of solutions with compact support for the equation $P(D)u = f$ when $f \in \mathcal{E}'(\mathbb{R}^n)$ (and as an application a refinement of the mean value theorem of Asgeirsson is proved) and again a classical result of Malgrange about approximation by means of exponental solutions (th. 7.3.6). Th. (7.3.8) describes the convex hull of the singular support of a distribution u by means of evaluation of its Fourier transform $\widehat{u}$, and an easy application is th. 7.3.9. Then one finds the result asserting that for every nonzero polynomial P in n variables there exist a fundamental solution E of $P(D)$ of finite order.

Section 7.4 "More general Fourier-Laplace transforms" begins by studying, for a given $u \in \mathcal{D}'(\mathbb{R}^n)$ the set $\Gamma_u = \{\eta \in \mathbb{R}^n, \exp\langle \cdot, n\rangle u \in \mathcal{S}'\}$ which is shown to be a convex set. Next, if the interior Γ^0 of Γ is not empty one can find an analytic function $\widehat{u}$ on $\mathbb{R}^n + i\Gamma_v$ which plays the role of a generalized Fourier-Laplace transform. (th. 7.4.2 and 7.4.3). These results are used to compute the Fourier transform of the advanced fundamental solution of the wave operator.

In section 7.5 the author gives a proof of Malgrange division theorem using the natural remark that decomposition of the Fourier transform $\widehat{\varphi}$ of a function φ of $\mathcal{S}$ by a partition of unity gives a representation of φ as a sum of entire analytic functions, an observation which is used to derive the theorem from the classical Weierstrass preparation theorem. In fact, the author makes a very careful study of this preparation theorem and actually proves more (th. 7.5.12, th. 7.5.135).

Section 7.6 "Fourier transforms of Gaussian functions" is of a more computational nature, but essential in the sequel. For the sake of completeness, the author also derives the central limit theorem (th. 7.6.7), even if it is outside the main topic. We also mention a discussion of the Airy function, and the construction of a two-sided fundamental solution of the Kolmogorov operator.

Sections 7.7 and 7.8 study, respectively, the method of stationary phase and oscillatory integrals. These are central topics, although their significance will really be seen in the forthcoming volumes. The treatment of the stationary phase method is very interesting because the author manages to treat also the case of a complex valued phase function in the same setting, and without use of the Morse lemma. The key results are th. 7.7.1, th. 7.7.5 and for the case with parameters, 7.7.6. The case of a complex phase is given by th. 7.7.12. Some beautiful applications of these results are given. The first is the asymptotic expansion of the Fourier transform $\widehat{u}(\tau\xi)$, $\|\xi| = 1$, $\tau > 1$ where $u = \operatorname{ad} S$ is a C_0^∞ density on a C^∞ hypersurface of total curvature $\kappa \neq 0$ (dS is the Euclidean surface measure) (th. 7.7.14), the second one is a classical estimate of the number of lattice points in convex sets in $\mathbb{R}^n$ (th. 7.7.16). There are also some results on the asymptotics of integrals of the form $\int ue^{i\omega f}dx$ where the integral is extended only on an open set X with C^∞ boundary ∂X.

Another point worth to be mentioned is the discussion of the asymptotic behaviour of $\int u(x)e^{i\omega f(x)}dx$, where f is real valued but has a degenerate critical point x_0. The integration is extended over the whole space, but parameters may be present. The author discusses the case when the Airy function turns up (th. 7.7.18) and gives also an extension of this result when there are several integration variables.

In the next section, oscillatory integrals with amplitudes of class $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ are studied, the relevant results being th. 7.8.3. As an application one finds the fundamental solution of the Cauchy problem for the wave operator. The last section of this chapter studies the $H_{(s)}$, L^p and Hölder spaces. Among the results to be found here, note th. 7.9.5 (Mikhlin-Hörmander) and as applications results on the regularity of solutions for elliptic or for constant coefficients operators.

The next two chapters, chapter VIII, "Spectral analysis of singularities" and chapter IX, "Hyperfunctions" reflect on one hand the progress made using what is now commonly known as microlocal analysis and the other a personal approach of the author towards hyperfunction theory. In §8.1 one finds the definition of the wave (WP) front set, and the calculation of WF for some distributions (see th. 8.15, 8.16, and 8.1.9). In §8.2 the possibility given by the microlocal analysis to define multiplications of distribution and composition with maps is developed, and some interesting examples are given. The third paragraph studies the propagation of singularities, and shows the great usefulness of considering things microlocally. Note th. 8.3.1 th. 8.3.7 about parametrices of operators with real principal type (with constant coefficients). In §8.4, the wave front set with respect to C^1 (where $L = \{L_k\}$ is an increasing sequence of positive numbers satisfying $k \le L_k$, $L_{k^n} \le CL_k$) is introduced, (and various properties for WF_L are derived (in analogy with WF)). In case $L = \{(k+1)\}_{k=1,2,\dots}$. C^L is the class of real analytic functions and the corresponding WF_L is denoted WF_A (the analytic wave front set). Now, there are several equivalent definitions of analytic wave front sets, and, even the author has obtained with this definition important result, it is not at all obvious that the one used here is the most natural or the most convenient, an opinion strengthened by the fact that, in order to prove the "melon eclaté" theorem of Kashiwara in ch. IX (th. 9.6.6) the author uses Bros-Iagolnitzer definition. In order to give an approach also suited to study hyperfunctions, in proving various results about WF_L the author uses an alternative method (i.e., not proceeding along lines analogous to what was done in the C^∞ case).

For this, to any distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ the author associates an analytic function U in the convex tube $\Omega = \{z \in \mathbb{C}^n \mid |\operatorname{Im} z| < 1\}$ so that u can be written as $u = \int_{|\omega|=1} V(\cdot + i\omega)d\omega$, i.e., a decomposition in waves along the unit sphere. The main result here is th. 8.4.11 that asserts that if $u \in \mathcal{S}'(\mathbb{R}^n)$ and $V = K * u$ (where

$$K = K(z) = (2\pi)^{-n} \int e^{i\langle z,\xi\rangle}/I(\xi)d\xi \ z \in \Omega, \text{ and } I(\xi) = \int_{|\omega|=1} e^{-\langle\omega,\xi\rangle}d\omega*$$

then U is analytic in Ω considered above, the boundary values $V(\cdot + 1\omega)$ are continuous functions of $\omega \in S^{n-1}$ with values in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\langle u,\varphi\rangle = \int \langle V(\cdot, +1\omega),\ \varphi\rangle d\omega \text{ for } \varphi \in \mathcal{S}', \qquad (**)$$

and U satisfies the estimate

$$|U(z)| \le C(1+|z|)^*(1-|\operatorname{Im} z|)^{-b},$$

$z \in \Omega$ and conversely if U satisfies the above estimate, (**) defines a distribution $u \in \mathcal{S}'$ with $U = K * u$, and $(\mathbb{R}^n \times S^{n-1}) \cap WF_L(u) = \{(x,\omega) \mid |\omega| = 1\ U$ is not in C^L at $x - i\omega\}$. Section 8.5 contains a detailed analysis of the methods of computations for WF_L; the introduction and usefulness of the notion of the normal set $N(F)$ of any closed set F is illustrated by th. 8.5.6 which asserts that $\overline{N}(\operatorname{supp} u) \subset WF_A(u)$ for any $u \in \mathcal{D}'(X)$; also very interesting is the geometric discussion about bicharacteristics strips (th. 8.5.9) and th. 8.5.11.

All these results are used in §8.4 in the study of WF_L for solutions of pde; we mention also Holmgren's theorem and a stronger unique continuation theorem (th. 8.6.6). Last section 8.6. is dedicated to the study of microhyperbolicity, the main result being th. 8.7.5.

The concluding chapter of this volume, Chapter IX is an introduction of hyperfunction theory with some significant applications. The approach given here is noncohomological and thus more elementary. The main feature is the introduction of what we might call the Hörmander indicatrix of an analytic functional (and thus of a general hyperfunction). More precisely, if K is a compact in $\mathbb{R}^n$ and $u \in A'(K)$ (analytic functionals on K) and if we denote by E the fundamental solution of the Laplacian in $\mathbb{R}^{n+1}$, let $P = \frac{\partial E}{\partial x_{n+1}}$

and $U = u_Y\ P(X-(y,0))$ (here $X=(x,x_{n+1})$ is a point in $\mathbb{R}^{n+1}$). This U is a harmonic function in $\mathbb{R}^{n+1}\setminus(K\times\{0\})$, odd as a function of x_{n+1} and it determines u completely.

There is an analogue of th. 8.4.11: If $u\in A'(\mathbb{R}^n)$ then $U(z)=K_*u(z)$ (where $K(z)$ (defined by (*)) corresponds to the decomposition of 0) is analytic in $Z=\{z|(\operatorname{Im} z|^2<1+|\operatorname{Re} z-t|^2,\ t\in\operatorname{supp} u\}$ and for a bounded neighbourhood X of $\operatorname{supp} u$ we have the representation:

$$u(\varphi)=\lim_{r\to 1}\int_{|\omega|=1|}\int_X U(x+ir\omega)\varphi(x)dx\,d\omega$$

for any $\varphi\in A$. Conversely, for any U analytic when $|\operatorname{Im} z|<1$ and any bounded open set X, if $\Sigma(U,X)=\{y\in\mathbb{R}^n,\ |y|=1$ such that U is analytic $x+iy$ for every $x\in\partial X\}$ then $U^x_y(\varphi)=\int_x U(x+iy)\varphi(x)dx$ is in $A'(X)$ if $y\in\mathbb{R}^n,\ |y|<1$;

$$U^X_\mu=\int U^X_\omega(\varphi)d\mu(\omega)=\lim_{r\to 1}\int_x\int U(x+ir\omega)\varphi(x)\,dx\,d\mu(\omega)$$

defines an element of $A'(X)$ for every measure $d\mu$ with support contained in $\Sigma(U,X)$. Now, the definition of $WF_A(u)$ can be given; $WF_A(u)=\{(x,\xi)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$ such that $U=K*u$ is not analytic at $x-\frac{i\xi}{|\xi|}$.

Reviewer: G. Gussi

For a scan of this review see the web version.

MSC:

35-02	Research exposition (monographs, survey articles) pertaining to partial differential equations
46-02	Research exposition (monographs, survey articles) pertaining to functional analysis
35G05	Linear higher-order PDEs
42A38	Fourier and Fourier-Stieltjes transforms and other transforms of Fourier type
46F05	Topological linear spaces of test functions, distributions and ultradistributions
35A05	General existence and uniqueness theorems (PDE) (MSC2000)
35B40	Asymptotic behavior of solutions to PDEs
46E35	Sobolev spaces and other spaces of "smooth" functions, embedding theorems, trace theorems
46F10	Operations with distributions and generalized functions
46F15	Hyperfunctions, analytic functionals
46F20	Distributions and ultradistributions as boundary values of analytic functions

Keywords:

distribution theory; harmonic analysis; regularization; quasi-analyticity; localization; Whitney extension theorem; convolution; subharmonic distributions; plurisubharmonic functions; Sobolev embedding theorem; Fourier-Laplace transformation; Gaussian functions; oscillatory integrals; spectral analysis of singularities; hyperfunctions; propagation of singularities; wave front set; continuation theorem; analytic Cauchy problem; global existence; microlocal existence