Li, Tien-Yien; **Yorke, James A.**

Period three implies chaos. (English) Zbl 0351.92021

Am. Math. Mon. 82, 985-992 (1975).

Let F be a continuous function of an interval J into itself. The period of a point in J is the least integer $k > 1$ for which $F^k(p) = p$. If p has period 3 then the relation $F^3(q) \leq q < F(q) < F^2(q)$ (or its reverse) is satisfied for q one of the points p, $F(p)$, or $F^2(p)$. The title of the paper derives from the theorem that if some point q in J has this Sysiphusian feature, "two steps forward, one giant step back", then F has periodic points of every period $K = 1, 2, 3, \dots$. Moreover, J contains an uncountable subset S devoid of asymptotically periodic points, such that

$$0 = \liminf |F^n(q) - F^n(r)| < \limsup |F^n(q) - F^n(r)|$$

for all $q \neq r$ in S. (a point is asymptotically periodic if $\lim |F^n(p) - F^n(q)| = 0$ for some periodic point p.) The proof is eminently accessible to the nonspecialist and is therefore of interest to anyone modeling the evolution of a single population parameter by a first order difference equation. The authors compare the logistic $x_{n+1} = F(x_n) = rx_n(1 - x_n/K)$ with a model of which, by contrast, $|dF(x)/dx| > 1$ wherever the derivative exists. For such a system no periodic point is stable, in the sense that $|F^k(q) - p| < |q - p|$ for all q in a neigborhood of a periodic point p of k. A brief survey of a theorem motivated by ergodic theory completes this fascinating paper.

Reviewer: G.K. Francis

For a scan of this review see the web version.

MSC:

92D25 Population dynamics (general)
39A10 Additive difference equations
54H20 Topological dynamics (MSC2010)
37N25 Dynamical systems in biology
37C25 Fixed points and periodic points of dynamical systems; fixed-point index theory, local dynamics

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