**Veselý, Libor**; **Duda, Jakub**; **Zajíček, Luděk**

**On d.c. functions and mappings.** (English) Zbl 1072.46025

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A real function $f$ defined on a convex subset $C$ of $\mathbb{R}^n$ is called a d.c. function if it can be written as the difference of two convex functions. In the case $n=1$, d.c. functions agree with the primitives of functions with locally bounded variations. In $\mathbb{R}^n$ they were considered for the first time by *A. D. Aleksandrov* [Dokl. Akad. Nauk SSSR, N. Ser. 72, 613–616 (1950; Zbl 0039.18003)] in connection with some problems of the geometry of surfaces in $\mathbb{R}^n$. An essential contribution to their study was done in the seminar paper of *Ph. Hartman* [Pac. J. Math. 9, 707–713 (1959; Zbl 0093.06401)]. A good survey of the properties of d.c. functions and of their applicability to some nonsmooth optimization problems was given by *J.–B. Hiriart–Urruty* [Lect. Notes Econ. Math. Syst. 256, 37–70 (1985; Zbl 0591.90073)]. Two of the authors of the present paper, *L. Veselý* and *L. Zajĭcek* [Diss. Math. 289 (1989; Zbl 0685.46027)] extended the notion of d.c. function to mappings between normed spaces.

A mapping $F: C \to Y$, where $X, Y$ are normed spaces and $C$ is an open convex subset of $X$, is called a d.c. mapping if there exists a continuous convex function $f: C \to \mathbb{R}$ such that $y^* \circ F + f$ is a continuous convex function for every $y^* \in Y^*$, $\|y^*\| = 1$. Any such function $f$ is called a control function for $F$. Other natural extensions are: (i) $F$ is called a weakly d.c. mapping if $y^* \circ F$ is a d.c. function for every $y^* \in Y^*$; (ii) if $Y$ is an ordered normed space, then $F$ is called an order d.c. mapping if it is the difference of two convex (with respect to the order in $Y$) operators from $C$ to $Y$. In the case $Y = \mathbb{R}^n$, all these three definitions agree, but in general they are different.

At the end of the last of the above mentioned works, there were posed ten problems. One of the aims of the present paper is to report on the stage of these problems. Four of them (problems 1, 2, 3 and 5) were solved in the negative (by counterexamples) by *J. Duda* [Commentat. Math. Univ. Carol. 42, No. 2, 281–297 (2001; Zbl 1053.47522)] and *E. Kopecká* and *J. Malý* [ibid. 31, No. 3, 501–510 (1990; Zbl 0714.46007)]. Another one, problem 6, is solved in the negative as well in the present paper; namely, the authors show that there exists a continuous quadratic mapping $F: \ell^3 \to \ell^\infty$ that is not locally d.c. The question was: if $F \circ \varphi$ is d.c. on $(0,1)$ for every d.c. mapping $\varphi: (0,1) \to C$, is then $F$ locally d.c. on $C$?

Problems 4, 7, 8, 9, 10 appear to be still open. The authors also give a characterization of Banach spaces $X$ such that $C^{1,1}$-mappings $F: C \to Y$ are d.c. mappings for every open convex $C \subset X$ and every normed space $Y$. They show that these are the Banach spaces that admit an equivalent uniformly smooth norm with modulus of convexity of power type 2. The Lipschitz properties of d.c. mappings and the problem of d.c. isomorphic classification of Banach spaces are briefly discussed as well.

Reviewer: Stefan Cobzaş (Cluj-Napoca)

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d.c. function; convexity; Banach spaces; weakly d.c. mapping; order d.c. mapping; Lipschitz property