Eskin, Alex; **Margulis, Gregory**; **Mozes, Shahar**
Quadratic forms of signature $(2,2)$ and eigenvalue spacings on rectangular 2-tori. (English)
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A quantitative version of the Oppenheim conjecture proved by Margulis states that for a nondegenerate indefinite quadratic form Q in n variables there exists a constant $\lambda_{Q,\Omega}$ such that for any interval (ab) as $T\to\infty$ $\mathrm{Vol}\{x\in\mathbb{R}^n : x\in T\Omega \text{ and } a\le Q(s)\le b\}\sim\lambda_{Q,\Omega(b-a)T^{n-2}}$, where $\Omega=\{v\in\mathbb{R}^n \mid \|v\|<\rho(v/\|v\|)\}$ and ρ is a continuous positive function on the sphere $\{v\in\mathbb{R}^n \mid \|v\|=1\}$. Eskin, Margulis and Mozes have shown that $N_{Q,\Omega}(a,b,T)\sim\lambda_{Q,\Omega}(b-a)T^{n-2}$ where Q is an indefinite quadratic form (not proportional to a rational form) of signature (p,q) with $p\ge 3$, $q\ge 1$, $n=p+q$ and $N_{Q,\Omega}(a,b,T)$ denotes the cardinality of the set $\{x\in\mathbb{Z}^n : x\in T\Omega \text{ and } a<Q(x)<b\}$. If the signature of Q is $(2,1)$ or $(2,2)$ then the above result fails. Whenever a form of signature $(2,2)$ has a rational isotropic subspace L then $L\cap T\Omega$ contains on the order of T^2 integral points x for which $Q(x)=0$, hence $N_{Q,\Omega}(-\varepsilon,\varepsilon,T)\ge cT^2$, independently of the choice of ε.

Thus to obtain an asymptotic formula in the signature $(2,2)$ case, we must exclude the contribution of the rational isotropic subspaces. The main result of this paper is as follows: Let Q be an indefinite quadratic form of signature $(2,2)$ which is not extremely well approximable by split forms then for any interval (a,b) as $T\to\infty$, $\widetilde{N}_{Q,\Omega}(a,b,T)\sim\lambda_{Q,\Omega}(b-a)T^2$ where $\widetilde{N}_{Q,\Omega}$ counts the points not contained in isotropic subspaces. It turns out that points belonging to a wider class of subspaces have to be treated separately. In order to estimate $N_{Q,\Omega}$ a transition to considering certain integrals on the space of unimodular lattices in $\mathbb{R}^n$ is made. This transition is based on the transitivity of the action of the orthogonal group $\mathrm{SO}(Q)$ on the level sets of the quadratic form Q. One also needs to have an estimate of the contribution of elements of lattices lying at the cusps of $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$.

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MSC:

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22E40	Discrete subgroups of Lie groups

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