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Walks in the quarter plane: Kreweras' algebraic model. (English) Zbl 1064.05010

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Summary: We consider planar lattice walks that start from (0,0), remain in the first quadrant $i, j \geq 0$, and are made of three types of steps: North-East, West and South. These walks are known to have remarkable enumerative and probabilistic properties:

• they are counted by nice numbers [*G. Kreweras*, Cahiers du B.U.R.O 6, 5–105 (1965)],

• the generating function of these numbers is algebraic [*I.M. Gessel*, J. Stat. Plann. Inference 14, 49–58 (1986; Zbl 0602.05006)],

• the stationary distribution of the corresponding Markov chain in the quadrant has an algebraic probability generating function [*L. Flatto* and *S. Hahn*, SIAM J. Appl. Math. 44, 1041–1053 (1984; Zbl 0554.90041)].

These results are not well understood, and have been established via complicated proofs. Here we give a uniform derivation of all of them, which is more elementary than those previously published. We then go further by computing the full law of the Markov chain. This helps to delimit the border of algebraicity: the associated probability generating function is no longer algebraic, unless a diagonal symmetry holds.

Our proofs are based on the solution of certain functional equations, which are very simple to establish. Finding purely combinatorial proofs remains an open problem.

MSC:

05A15 Exact enumeration problems, generating functions
60J10 Markov chains (discrete-time Markov processes on discrete state spaces)

Keywords:

lattice walks; enumeration; algebraic generating functions; Markov chains in the quarter plane

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