Hori, Kentaro; **Katz, Sheldon**; **Klemm, Albrecht**; **Pandharipande, Rahul**; **Thomas, Richard**; **Vafa, Cumrun**; **Vakil, Ravi**; **Zaslow, Eric**
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The book under review collects mathematical and physical materials presented at the Clay Mathematics Institute Spring School on Mirror Symmetry in 2000 at Pine Manor College, Brookline, Massachusetts. The book is purported to make a bridge between mathematicians and physicists working on mirror symmetry developing mutually understandable language and mutual respect. Accordingly, almost half of the book is devoted to introductory materials in mathematics and physics surrounding mirror symmetry with emphasis on the interconnection between the two fields.

Mirror symmetry is a phenomenon, commonly known as "duality". The idea of mirror symmetry has originated in physics, e.g., T-duality; the theory has exploded onto the mathematical scene. In the last decades, this interaction between mathematics and physics has been phenomenal.

Calabi-Yau manifolds have emerged as interesting geometries for string propagation, tied with $N = 2$ supersymmetry. This has led to the formulation of the mirror symmetry conjecture that two seemingly different Calabi-Yau manifolds of dimension three would give rise to the isomorphic physical theory. This mirror symmetry prediction has been established for a large class of Calabi-Yau threefolds. Also mirror symmetry enables us to compute the number of rational curves on a Calabi-Yau threefold in terms of Hodge structure for the mirror Calabi-Yau threefold.

The notion of topological mirror symmetry has been introduced, among other reasons, to count the number of holomorphic maps from higher genus curves to Calabi-Yau threefolds. This is accommodated by introducing D-branes. The mirror symmetry conjecture is also formulated as the statement of the equivalence of the derived category and the Fukaya category. Toric geometry is used to construct a large class of mirror pairs of Calabi-Yau manifolds.

The book consists of 40 chapters, divided into five parts. Part I covers chapters 1 to 7 and deals with mathematical preliminaries. Part II covers chapters 8 to 19, and aims to give physical preliminaries. Part III gives a "physics proof" to the mirror symmetry conjecture, and part IV (covering chapters 21 to 30) presents a mathematical proof to the mirror symmetry conjecture. Finally part V covers chapters 31 to 40 and is devoted to more advanced topics.

Here is a more detailed description of each chapter. Each chapter carefully chooses materials which are essential for understanding mirror symmetry, and the treatments are often centered around specific examples.

Chapter 1 (Differential geometry): This chapter gives a review on the basics of differential geometry, including manifolds and submanifolds, vector bundles, differential forms and integration.

Chapter 2 (Algebraic geometry): This chapter outlines the very basic constructions of algebraic geometry, including projective spaces, toric geometry, the hyperplane line bundles, sheaves and cohomology, and divisors.

Chapter 3 (Differential and algebraic topology): This chapter contains various cohomology theories, and discussions on characteristic classes, and their applications.

Chapter 4 (Equivariant cohomology and fixed-point theorems): This chapter contains a synopsis of various theorems concerning the localization of calculations of fixed points of diffeomorphisms, zeros of vector fields or sections, or fixed points of group actions. The motivation stems from the calculation of Gromov-Witten invariants.

Chapter 5 (Complex and Kähler geometry): This chapter discusses the basics of complex geometry and Kähler metrics. The discussions are motived by the fact that manifolds with Kähler metric admit the $N = 2$ supersymmetric sigma models crucial for formulating mirror symmetry. Also Calabi-Yau conditions are discussed.

Chapter 6 (Calabi-Yau manifolds and their moduli): This chapter discusses deformations of a complex structure, the moduli space of the complex structure of a Calabi-Yau manifold, and singularities and their resolutions. The discussions are centered around the example of a quintic Calabi-Yau threefold and its mirror.

Chapter 7 (Toric geometry for string theory): This chapter gives a thorough treatment on toric geometry for string theory, and then interpreting mirror symmetry in terms of toric geometry explaining the construction due to Batyrev.

Chapter 8 (What is a QFT?): This chapter serves as a "practical guide" introduction to the part 2 (chapters 8 to 19), Physics Preliminaries. The main aim is to introduce a quantum field theory through examples. Choose a manifold M of dimension d (with/without boundaries). Then choose objects on M, called fields. The operation of integration over the fields is called the path-integral.

Chapter 9 (QFT in $d = 0$): This chapter discusses zero-dimensional quantum field theories, i.e., when a manifold M is a point. Feynman diagrams, fermions, localization, partition functions and zero-dimensional Landau-Ginzburg theory are discussed in supersymmetric quantum field theories.

Chapter 10 (QFT in dimension 1: quantum mechanics): This chapter discusses one-dimensional quantum field theories (quantum mechanics) in the context of supersymmetry. In particular, supersymmetric quantum mechanical systems corresponding to maps from one-dimensional space to target spaces (which are Riemannian or Kähler manifolds). These are known as sigma models. The main focus is on supersymmetric ground states.

Chapter 11 (Free quantum field theories in $1 + 1$ dimensions): This chapter discusses QFT in two dimensions, focusing mainly on free QFT. Here free refers to the fact that the action is quadratic in the field variables. In supersymmetric theories, important quantities are determined using quadratic approximation of the theory, which illustrates the pivotal role of free field theories. Also the sigma model with a circle of radius R as a target is an example of free theory, and a T-duality between the sigma model of radius R and that of radius $1/R$ is exhibited with free theory.

Chapter 12 ($N = (2, 2)$ supersymmetry): This chapter discusses mainly supersymmetric field theories in $1 + 1$ dimensions with four real supercharges, two with positive chirality and two with negative chirality. This is called $N = (2, 2)$ supersymmetry, and mirror symmetry is described for two $N = (2, 2)$ supersymmetric quantum field theories.

Chapter 13 (Nonlinear sigma models and Landau-Ginzburg models): The nonlinear sigma model on a compact connected Kähler manifold M with the superprepotential set to zero is considered aiming toward finding the number of supersymmetric ground states. The conclusion is that the supersymmetric ground states are in one-to-one correspondence with the harmonic forms on M. Also the T-duality is shown to be a mirror symmetry analyzing supersymmetric sigma model on T^2.

Chapter 14 (Renormalization group flow): This chapter discusses the most important aspects of quantum field theory, namely, the rescaling of the metric on the manifold over which the quantum field theory is defined. The target space metric is seen to change as a function of the scale. However, the superpotential in a Landau-Ginzburg model does not depend on the scale. This is the famous non-renormalization theorem, and its proof and its generalizations are presented in this chapter.

Chapter 15 (Linear sigma models): A class of supersymmetric gauge theories in $1 + 1$ dimensions, called linear sigma models, is studied, and a global description of linear sigma models is provided. This model is the essential tool for the proof of mirror symmetry.

Chapter 16 (Chiral rings and topological field theory): The chiral rings of $(2, 2)$ supersymmetric field theories are studied. Two important aspects of $(2, 2)$ supersymmetry theories are: the structure of the vacuum states, and the structure of chiral fields and rings. Modified $(2, 2)$ theories, called topological field theories are introduced. Topological fields theories coincide with ordinary $(2, 2)$ theories on flat worldsheets, but differ from them on curved Riemann surfaces (known as "topological twisting") in a way of preserving half of the supersymmetries. Several classes of twisted theories are studied, and computations of the chiral rings are carried out on some examples.

Chapter 17 (Chiral rings and the geometry of the vacuum bundle): The operator/state correspondence in QFT relates the vacuum states and the chiral fields and rings. However, there seems to be more information in the chiral rings than in the vacuum states. This chapter explores if there is any further information in the structure of the ground states that encodes the structure of the chiral ring. This sought after information is encoded in how vacuum states vary in the full Hilbert space of the theory when the superpotential parameters are varied. The connection and metric on this vacuum bundle, and their relation to chiral rings, are described by the tt^* equations.

Chapter 18 (BPS solitons in $N = 2$ Landau-Ginzburg theories): This chapter discusses another example of physical quantities that depend only on the superpotential terms. The spectrum of (BPS) solitons is such an example. This connects the study of Landau-Ginzburg theories to the Picard-Lefschetz theory of vanishing cycles. Also the relation between tt^* geometry and BPS solutions is discussed.

Chapter 19 (D-branes): This chapter introduces one important piece of ingredients in the mirror symmetry theory, namely, D-branes. What are D-branes? D-branes appear when one considers sigma models for maps from Riemann surfaces with boundaries, with some natural boundary conditions. A boundary is mapped to the so-called D-brane in the target space. When the target space is a circle with radius R, the T-duality symmetry (which relates $R \to 1/R$ symmetry) induces an action on D-branes exchanging D0-branes (associated to the Dirichlet boundary condition) with D1-branes (associated to Neumann boundary conditions). The mirror symmetry theory is interpreted in terms of D-branes.

Chapter 20 (proof of mirror symmetry): This chapter is rather short, and it presents a physical proof of mirror symmetry due to Hori and Vafa, first clarifying what is meant by "Proof". Then the proof is divided into three steps. Here "proving mirror symmetry" is referred to establishing the equivalence of two theories: a gauged linear sigma model, and a Landau-Ginzburg theory with a certain superprepotential, W, up to D-term variations. When certain Landau-Ginzburg theories can be viewed as Calabi-Yau sigma models, in which case the B-ring of Landau-Ginzburg theory maps to the B-model topological ring of the Calabi-Yau. In this case, mirror symmetry maps the A-model topological amplitudes in one Calabi-Yau M to the B-model topological amplitudes of another Calabi-Yau $\widetilde{M}$. This gives rise to a relation between the Hodge numbers of the Calabi-Yau: $h^{p,q}(M) = h^{d-p,q}(\widetilde{M})$ where d is the complex dimension of M and $\widetilde{M}$. The proof presented here is more general, i.e., it addresses also the cases where the Landau-Ginzburg theories do not correspond to sigma models on Calabi-Yau manifolds (for instance, mirrors of Fano varieties). The basic ingredient in the proof is a formulation of the sigma model in the context of the gauged linear sigma model and application of T-duality ($R \to 1/R$ duality) to the charged fields of the gauged linear sigma model. Then mirrors are constructed for toric varieties, and then this is generalized to hypersurfaces (or complete intersections) in toric varieties.

Chapter 21 (Introduction and overview): Chapters 21 to 30 introduce Gromov-Witten theory to both mathematicians and physicists. This chapter gives an overview of the forthcoming chapters. Necessary notations and conventions are set up here. For instance, Σ and $\overline{\Sigma}$ stand for, respectively, nodal curves and its normalization, $\mathcal{M}_g$, $\overline{\mathcal{M}}_g$, $\mathcal{M}_{g,n}$, $\overline{\mathcal{M}}_{g,n}$ moduli spaces of curves, $\overline{\mathcal{M}}_{g,n}(X, \beta)$ moduli space of stable maps. Also the concept of Deligne-Mumford stacks is introduced.

Chapter 22 (Complex curves (nonsingular and nodal)): Topological surfaces are introduced as differentiable manifolds of dimension two that are oriented, compact and connected. The genus is introduced. Riemannian structure, conformal structure, almost complex structure, complex structure and finally algebraic structure on topological surfaces are introduced, and relations among them are briefly discussed. Also the Riemann-Roch theorem is stated. Complex curves with the simplest singularities, namely, nodal curves, their normalizations, and differentials on nodal curves are described. Also an interpretation of nodal curves in terms of dual graphs is presented.

Chapter 23 (Moduli spaces of curves): The moduli space $\mathcal{M}_g$ of non-singular Riemann surfaces is defined. If $g > 1$, $\mathcal{M}_g$ is a nonsingular Deligne-Mumford stack, and furthermore, if $g > 2$, it is actually an orbifold. $\mathcal{M}_g$ is compactified via the Deligne-Mumford compactification, and the compactification is denoted by $\overline{\mathcal{M}}_g$. It includes special nodal curves, the so-called stable curves. Further, compactifying the moduli space $\mathcal{M}_{g,n}$ of n-pointed genus g Riemann surfaces, one obtains a compact nonsingular moduli space $\overline{\mathcal{M}}_{g,n}$, whose dimension is $3g - 3 + n$. The boundary $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is stratified by dual graph type.

Chapter 24 (Moduli spaces $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of stable maps): The moduli space of stable maps is described together with some of its properties. Let X be a nonsingular projective variety. A stable map is a map f from a pointed nodal curve to X such that every genus 0 contracted component of Σ has at least three special points, and every genus 1 contracted component has at least one special point. Seven properties of the moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ are discussed, including compactness of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ and the existence of "evaluation maps ev, forgetful morphisms, a universal map". Furthermore, deformation theory of stable maps is very briefly discussed.

Chapter 25 (Cohomology classes on $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(X, \beta)$): This chapter describes certain naturally defined cohomology classes on the moduli spaces of stable maps. These include classes pulled back from

X, the classes defined by the tautological line bundles, and the classes defined by the Hodge bundle. It is shown that the moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ is nonsingular and equi-dimensional. Also a recursion for rational plane curves is discussed here. Let N_d denote the number of rational curves to $\mathbb{P}^2$ through $3d-1$ general points in $\mathbb{P}^2$. For $d > 1$, there is a formula for N_d, and the proof is presented here.

Chapter 26 (The virtual fundamental class, Gromov-Witten invariants, and descendant invariants): The virtual fundamental class is discussed, and then Gromov-Witten and descendant invariants are defined. The space of stable maps carries a virtual fundamental class, denoted $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{vir}}$, which lies in the expected dimension. The full construction is not presented here but in three special cases, a simple interpretation of the virtual class is presented. In the case the moduli space is unobstructed, the virtual fundamental class is the ordinary fundamental class. When the moduli space is nonsingular, the virtual fundamental class is the Euler class. In the case $g = 0$ and X is a hypersurface, there is a canonical homology class which describes the virtual fundamental class, which involves $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$. Relations among these three special cases are worked out, as well as Witten's formula. The virtual fundamental class is paired against the cohomology classes to define the Gromov-Witten invariants. Given cohomology classes $\gamma_1, \cdots, \gamma_n \in H^*(X)$, the Gromov-Witten invariant is defined by

$$\langle \gamma_1, \cdots, \gamma_n \rangle_{g,\beta}^X := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{vir}}} ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n).$$

A generalization of the Gromov-Witten invariants, the descendant invariants are defined, which couple Gromov-Witten invariants with topological gravity. The three equations, namely, string, dilation and divisor equations hold for descendant invariants. Also in the case of genus 0, descendant invariants can be uniquely reconstructed from Gromov-Witten invariants. This chapter also discusses the quantum cohomology ring, and small quantum cohomology ring.

Chapter 27 (Localization on the moduli space of maps): The techniques of torus localization on the moduli spaces of stable maps to $\mathbb{P}^m$ is introduced. The torus action on $\mathbb{P}^m$ lifts to an action on the space of maps, and integrals over the moduli space can be reduced via the localization formula to integrals over the space of maps fixed by the torus, which are much simpler. These methods are applied to $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ identifying the fixed loci with decorated graphs, and then compute their normal bundles. As an application, localization is used to prove the Aspinwall-Morrison formula for the contribution of genus 0 covers of $\mathbb{P}^1 \subset X$ where X is Calabi-Yau. The localization techniques are discussed in higher genus in the context of the virtual class, "virtual localization". Then the full multiple cover formula for $\mathbb{P}^1$ is obtained and its proof is presented. The multiple cover contributions are determined in higher genus by $C(1, d) = 1/(12d)$ for $g = 1$, and $C(g, d) = \frac{|B_{2g}|\cdot d^{2g-3}}{2g\cdot(2g-2)!} = |\chi(\mathcal{M}_g)|\frac{d^{2g-3}}{(2g-3)!}$ for $g \geq 2$. Here B_{2g} is the $2g$-th Bernoulli number and $\chi(\mathcal{M}_g)$ is the orbifold Euler characteristic of $\mathcal{M}_g$.

Chapter 28 (The fundamental solution of the quantum differential equation): This chapter studies a solution of a differential equation arising in mirror symmetry and quantum cohomology. The quantum differential equation is defined and its fundamental solution is obtained. These objects take simpler forms if one passes to the small quantum differential equation and its solution.

Chapter 29 (The mirror conjecture for hypersurfaces. I: The Fano case): The mirror conjecture for a quintic threefold $X \subset \mathbb{P}^4$ is formulated, and the proof is provided along the line of Givental's method. Let X be a hypersurface in $\mathbb{P}^m$ of degree $\ell \leq m+1$. The "correlator" $S_X(t)$ is obtained from the fundamental solution of the quantum differential equation for X, and encapsulate Gromov-Witten and descendant invariants. It is related to a certain hypergeometric series $S_X^*(t)$, which arises as a solution to the Picard-Fuchs differential equation. The precise relationship is divided into three cases : (i) Fano index > 1 case, (ii) Fano index 1 case, and (iii) Calabi-Yau case. This chapter discusses the first two cases. The mirror conjecture is a relation between $S_X(t)$ and $S_X^*(t)$. In case (i), $S_X(t) = S_X^*(t)$, and in case (ii), $S_X(t) = e^{-m!e^t} S_X^*(t)$, and proofs are presented here. The Clemens conjecture is stated for a generic quintic threefold.

Chapter 30 (The mirror conjecture for hypersurfaces. II: The Calabi-Yau case): This chapter presents a proof of the mirror conjecture (that $S_X(t)$ and $S_X^*(t)$ are related by mirror transformation) in the Calabi-Yau case, which is the most difficult and interesting case. Again the exposition follows Givental's idea. First correlator recursions (e.g., z-recursion) are written down, though they involve functions which are not yet determined. Givental goes around this problem imposing additional conditions on the correlators. This is accommodated by the polynomiality constraint. A geometric construction is carried out to understand the polynomiality constraint on the correlator $\Phi(z, e^t)$, or the double correlator $\Phi^Y(z, e^t)$ (where Y is a correlator). A necessary and sufficient condition is found for a correlator to satisfy Givental's polynomiality condition. A special class of correlators, called correlators of class $\mathcal{P}$, is introduced. A set of correlators is in class $\mathcal{P}$ if it satisfies (i) the rationality and regularity conditions, (ii) a special recursive relation, and (iii) polynomiality condition. For those correlators in class $\mathcal{P}$, a uniqueness result is proved, which is essential for a proof of mirror symmetry conjecture. Then an explicit transformation is found between $S^*_X(t, h = 1)$ and $S_X(t, h = 1)$ that yields the mirror prediction in the quintic threefold case.

The next nine chapters are devoted to advanced topics, discussing topological strings at higher genus and holomorphic anomaly, applications of mirror symmetry, D-branes, Kontsevich's mirror symmetry conjecture, Fukaya category, the derived categories, and open strings, among other results.

Chapter 31 (Topological strings): In the two-dimensional QFT theories, one fixes the worldsheet geometry, i.e., fix a Riemann surface. The idea of a modified theory is to vary Riemann surfaces. This chapter discusses one such generalized theory, known as topological string theory, first introduced by Witten. The modified theory can be viewed as a special type of "bosonic string" theory. In this theory, the topological sigma models are coupled with worldsheet gravity. What does it mean to integrate over worldsheet geometries in the context of topological sigma models? A topological string theory for Calabi-Yau threefolds is especially nice and defined here. Let $\mathcal{M}_g$ be the moduli space of Riemann surface of genus g. For $g > 1$, the genus g topological string amplitude F_g is defined involving the measure on $\mathcal{M}_g$. (For $g = 1$, some modification is required.) F_g should be viewed as a section of a bundle over it, rather than a function on the moduli space of Calabi-Yau manifolds. Also F_g is not holomorphic. To measure how it fails to be holomorphic, the anomaly equation is introduced.

Chapter 32 (Topological string amplitudes): This chapter gives a reinterpretation of the topological string amplitude from the target space point of view. This requires lots more background in string theory, and this chapter purports to do that. There are two types of superstring theories of closed strings, namely Type IIA and Type IIB theories. Worldsheet conformal invariance imposes that the target space should be ten-dimensional, a physically interesting case is $\mathbb{R}^{3,1} \times X^{(6)}$: the product of four-dimensional Minkowski space with a six-dimensional Calabi-Yau threefold. If X is a generic Calabi-Yau threefold X with SU(3), holonomy corresponds to $N = 2$ supersymmetry in four dimensions. What topological string computes in connection with target space physics questions? Let $F_g(t_i)$ denote the genus g topological amplitude. In the Type IIA-model, F_g will be the partition function on X and the t_i parameterize the Kähler class of X and in the Type IIB-model, the t_i parameterize complex moduli of X. Mirror symmetry relates the A-model to the corresponding M-model in the mirror Calabi-Yau, and the theory of Type IIA superstring on X is equivalent to the theory of Type IIB superstring on the mirror Calabi-Yau. (This is what is commonly referred to "mirror symmetry" in the mathematical literature.) To define topological strings on Riemann surfaces with boundaries, with a choice of boundary condition, the concept of D-branes is introduced. The topological string amplitude is determined by the Gopakumar-Vafa invariants n^Q_r (which are all integers).

Chapter 33 (Mathematical formulation of Gopakumar-Vafa invariants): The Gopakumar-Vafa invariants completely determine the topological amplitudes. This chapter presents the mathematical framework that computes the Gopakumar-Vafa invariants. Recall that Gopakumar-Vafa invariants n^Q_r capture the $\mathrm{SU}(2)_L$ content of the number of wrapped BPS D2-branes with charge $Q \in H_2(X, \mathbb{Z})$. For a generic Calabi-Yau threefold X, there is a formula $F_g = \sum_{\beta \in H_2(X,\mathbb{Z})} N^g_\beta q^\beta$, where $N^g_\beta \in \mathbb{Q}$ are the Gromov-Witten invariants and are given by the degree of the virtual fundamental class $\deg[\overline{\mathcal{M}}_{g,0}(X, \beta)]^{\mathrm{virt}}$. The data of the N^g_β is encoded in the BPS invariants n^r_β (or n^Q_β). The goal of this chapter is to understand these invariants. A number of examples on how to calculate N^r_β are discussed, and mathematical arguments are presented for the definition of n^g, which involves Hilbert schemes.

Chapter 34 (Multiple covers, integrality, and Gopakumar-Vafa invariants): This chapter discusses the integrality conjectures of Gopakumar-Vafa and a generalized integrality conjecture for Gromov-Witten invariants for Calabi-Yau threefolds. The conjecture is that the Gromov-Witten invariants n^0_β are integers for all Calabi-Yau threefolds X and curve class $\beta \neq 0$. A geometric interpretation of n^0_β is proposed by Gopakumar and Vafa in a more general setting. First Gopakumar and Vafa define the Gopakumar-Vafa invariants n^g_β for each genus g and each nonzero curve class $\beta \in H_2(X, \mathbb{Z})$ by an explicit formula, and then they conjecture that n^g_β are integers for all Calabi-Yau threefolds X, genus $g \geq 0$, and curve classes $\beta \neq 0$. The Gopakumar-Vafa invariants n^g_β arise as BPS state counts in Type IIA string theory. The background of the Gopakumar-Vafa conjecture is presented here together with the further generalization to vector-valued invariants by Pandharripande.

Chapter 35 (Mirror symmetry at higher genus): This chapter discusses how to use mirror symmetry to count holomorphic maps from higher genus curves to Calabi-Yau threefolds. The genus 1 closed string amplitude on the two-dimensional target space T^2 is calculated. This is compared with the Ray-Singer torsion. The B-model calculation of closed string higher-genus amplitudes on Calabi-Yau threefolds uses the holomorphic anomaly equations. The mirror map is introduced to identify the A-model and B-model topological string amplitudes. Holomorphic anomaly equations are further studied here, and these are solved for a number of examples. The topological open string annulus amplitude on T^2 is also discussed.

Chapter 36 (Some applications of mirror symmetry): This chapter discusses a couple of applications of mirror symmetry and topological string amplitudes. The first application is the so-called geometric engineering of gauge theories. Singularities of the target space of string theory have an interpretation as giving rise to gauge theories in target space, and mirror symmetry allows one to compute certain quantities in the gauge theory. The idea of identifying string propagation in geometric singularities with certain gauge theories is known as geometric engineering of gauge theories. The exposition on this application is focused on specific examples. Starting with ADE singularities of $K3$ surfaces and ADE gauge groups, Seiberg-Witten theory is illustrated for $N = 2$ supersymmetric SU(2) gauge theory in four dimensions. The gauge coupling $\tau(a)$ is the sum of a perturbative part and a non-perturbative part, and moreover, τ is related to the genus 0 prepotential of the topological string and hence can be computed using mirror symmetry. Another application involves studying the large N limit of gauge theories. This is a Chern-Simon theory on topological strings in the presence of D-branes. The SU(N) Chern-Simon theory on S^3 is equivalent to the closed topological string A-model on the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{CP}^1$.

Chapter 37 (Aspects of mirror symmetry and D-branes): This chapter discusses how mirror symmetry acts on D-branes. D-branes are incorporated into mirror symmetry, and the D-brane correspondence under mirror symmetry leads to the Strominger-Yau-Zaslow conjecture about the structure of Calabi-Yau manifolds with mirror symmetry (special Lagrangian fibrations) and how to find the mirror manifold geometrically. The mirror symmetry action on a D-brane is described in detail for the elliptic curve case. A mathematical characterization of D-branes is presented, e.g., D-branes are viewed mathematically by Kontsevich as objects in a category and the D-brane correspondence (mirror symmetry) is interpreted as an equivalence of two categories. Let $(M, \widetilde{M})$ be a mirror pair of Calabi-Yau threefolds. The categories involved in Kontsevich's mirror scenario are the derived category $\mathcal{D}^b(M)$ and the Fukaya A^∞ category $\mathcal{F}^0(\widetilde{M})$. Kontsevich's conjecture states that $(M, \widetilde{M})$ form a mirror pair if and only if $\mathcal{D}^b(M) \simeq \mathcal{F}^0(\widetilde{M})$. The categorical equivalence is proved in detail in the case of elliptic curves or a torus.

Chapter 38 (More on the mathematics of D-branes: bundles, derived categories, and Lagrangians): In this chapter, the derived category is explained. Kontsevich's prediction is that the derived category is a very strong invariant of a complex variety (in fact, stronger than D-branes, cohomology theory or K-theory), and the whole B-model string theory should be recovered from it. On the A-model side, D-branes are given by special Lagrangians and this is explained in detail here.

Chapter 39 (Boundary $N = 2$ theories): This chapter studies quantum field theory on $1 + 1$-dimensional manifolds with boundary. Such a system appears as the theory on the worldsheet of an open string. Topics studied here are free field theories with linear boundary conditions, boundary states and their properties, and the axial anomaly induced from boundary conditions, among others. The spectrum of supersymmetric states of various open string systems is determined.

Chapter 40 (References): This chapter lists references. In particular, for part V, "Advanced topics", the list of references is extensive.

In conclusion, the book covers tremendous amounts of pedagogical materials in mathematics and physics as well as the recent developments on mirror symmetry. It can be regarded as the most updated and most comprehensive book on the subject. It is highly recommended for graduate students, and young as well as established researchers in mathematics and physics who wish to learn about mirror symmetry.

Reviewer: Noriko Yui (Kingston)

MSC:

14J32 Calabi-Yau manifolds (algebro-geometric aspects)
14-02 Research exposition (monographs, survey articles) pertaining to algebraic geometry
81-02 Research exposition (monographs, survey articles) pertaining to quantum theory
81T30 String and superstring theories; other extended objects (e.g., branes) in quantum field theory
14N10 Enumerative problems (combinatorial problems) in algebraic geometry
14N35 Gromov-Witten invariants, quantum cohomology, Gopakumar-Vafa invariants, Donaldson-Thomas invariants (algebro-geometric aspects)
32G81 Applications of deformations of analytic structures to the sciences
83E30 String and superstring theories in gravitational theory
81T60 Supersymmetric field theories in quantum mechanics
14J81 Relationships with physics
32Q25 Calabi-Yau theory (complex-analytic aspects)
81T45 Topological field theories in quantum mechanics
57R56 Topological quantum field theories (aspects of differential topology)

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