Karlsen, Kenneth H.; Risebro, Nils H.; Towers, John D.

**On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient.** (English) Zbl 1015.35049

Electron. J. Differ. Equ. 2002, Paper No. 93, 23 p. (2002).

This paper concerns the following Cauchy problem: $\partial_t u + \partial_x(\gamma(x)f(u)) = \partial_x^2 A(u)$ in $\Pi_T = \mathbb{R}\times(0,T)$, $u(x,0) = u_0(x)$. The coefficient $\gamma(x)$ appearing in the transport part satisfies suitable conditions and may be discontinuous. The function $f(s)$ belongs to $C^2([0,1])$, satisfies $f(0) = f(1) = 0$ and there is no subinterval of $[0,1]$ on which it is linear. The function $A(s)$ belongs to $C^2([0,1])$, satisfies $A'(s)\geq 0$ and $A(0) = 0$. This weak parabolicity is general enough to include the hyperbolic conservation law $\partial_t u + \partial_x(\gamma(x)f(u)) = 0$. The initial data $u_0(x)\in[0,1]$ belongs to $L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$. If $A'(s) = 0$ on an interval, solutions may be discontinuous and they are not uniquely determined by the initial data: an entropy condition is imposed in this case. To prove existence of a weak solution the authors introduce a family of approximation problems where $\gamma(x)$ is replaced by a smooth coefficient $\gamma_\varepsilon(x)$ and $A(u)$ is replaced by a strictly increasing function $A_\varepsilon(u)$. Then they prove that a sequence of solutions $u_\varepsilon(x)$ converges in the $L^1$ norm to a solution of the previous problem. Furthermore, a subsequence of $A_\varepsilon(u_\varepsilon)$ converges uniformly on compact sets to a Hölder continuous function which coincides with $A(u)$ almost everywhere. If $\gamma(x)$ is discontinuous, the total variation of $u_\varepsilon$ cannot be bounded uniformly with respect to $\varepsilon$, and the standard BV compactness argument cannot be applied. To get around this difficulty, they establish a strong compactness of the diffusion function $A_\varepsilon(u_\varepsilon)$ as well as the total flux $\gamma_\varepsilon(x)f(u_\varepsilon) - \partial_x A_\varepsilon(u_\varepsilon)$ and apply the compensated compactness method. Also the purely hyperbolic case $A' = 0$ is discussed by derivation of strong convergence via some a priori energy estimates that may have independ interest.

Reviewer: Giovanni Porru (Cagliari)

**MSC:**

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| 35K65 | Degenerate parabolic equations |
| 35R05 | PDEs with low regular coefficients and/or low regular data |
| 35L80 | Degenerate hyperbolic equations |
| 35D05 | Existence of generalized solutions of PDE (MSC2000) |



**Keywords:**

approximation problems; nonconvex flux; viscosity method; a priori estimates; compensated compactness

**Full Text:** EMIS EuDML