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Torus actions and their applications in topology and combinatorics. (English) Zbl 1012.52021

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Today it is well known that the theory of toric varieties is intimately linked to the theory of convex polytopes, see the monograph by *G. Ewald* [‘Combinatorial convexity and algebraic geometry’, Springer (1996; Zbl 0869.52001)]. What makes this connection interesting is the fact that there is a number of non-trivial results in both fields which seem to be proved best with the techniques of the respective other. Starting out with a paper by *M. W. Davis* and *T. Januszkiewicz* [Duke Math. J. 62, No. 2, 417-451 (1991; Zbl 0733.52006)], the last decade saw a raising interest in topological generalizations.

The titles of the chapters of the research monograph under review are: 1. Polytopes, 2. Topology and combinatorics of simplicial complexes, 3. Commutative and homological algebra of simplicial complexes, 4. Cubical complexes, 5. Toric and quasitoric manifolds, 6. Moment-angle complexes, 7. Cohomology of moment-angle complexes and combinatorial triangulated manifolds, 8. Cohomology rings of subspace arrangement complements.

The first four chapters contain introductory material, which essentially sums up the background of the g-theorem due to *R. P. Stanley* [Adv. Math. 35, 236-238 (1980; Zbl 0427.52006)], *L. J. Billera* and *C. W. Lee* [Bull. Am. Math. Soc., New Ser. 2, 181-185 (1980; Zbl 0431.52009)]. The story begins with convex polytopes and some of their combinatorial properties. Key topics in Chapter 1 are f-vectors of (simplicial) polytopes and *P. McMullens*’s Upper Bound Theorem [Mathematika, Lond. 17, 179-184 (1970; Zbl 0217.46703)]. Relaxing the convexity conditions in Chapter 2 means to pass from simplicial polytopes to simplicial spheres and more general triangulated manifolds. Combinatorial topology is tied to commutative algebra: To each finite simplicial complex Δ (and some field k, say, of characteristic 0) one can construct a finitely generated algebra $k(\Delta)$, the Stanley-Reisner ring of Δ. By investigating the properties of $k(\Delta)$ *R. P. Stanley* [Stud. Appl. Math. 54, 135-142 (1975; Zbl 0308.52009)] could generalize the Upper Bound Theorem from polytopes to spheres. At the end of Chapter 3 the reader finds recent results from one line of research which seeks to extend Stanley’s theorem to wider classes of simplicial complexes by applying more refined methods from commutative and homological algebra.

The main theme of the book shows up in the fifth chapter. In Section 5.1 the authors sketch Stanley’s famous proof of one part of the g-theorem: The f-vector of a simplicial polytope P satisfies certain inequalities, which are induced from algebraic properties of $k(\partial P)$. One driving force behind current research in the area is the notoriously open g-conjecture, which states that the set of f-vectors of all simplicial spheres is the same as the set of f-vectors of all simplicial polytopes. The key step in Stanley’s proof relies on the Hard Lefschetz Theorem for toric varieties, which says that the vector of rational Betti numbers of the toric variety M_{P^*} associated to the dual P^* of P is symmetric. Topologically, one can obtain M_{P^*} as the quotient of the product of P^* with the d-torus $(\mathbb{R}/\mathbb{Z})^d$ by an equivalence relation which is defined by the combinatorics of P^*. This construction was generalized by Davis and Januszkiewicz [loc. cit.] to define quasi-toric manifolds in order to further the understanding of torus actions on smooth manifolds. Again a simple polytope P^* is the essential ingredient, but this time the resulting manifold is not always determined by the combinatorics of P^*.

The rest of the fifth chapter is devoted to the cohomological properties of quasi-toric manifolds and their relationship to the theory of cobordisms. A variation in the construction of quasi-toric manifolds, also due to Davis and Januszkiewicz [loc. cit.], leads to the moment-angle manifold Z_{P^*} of a simple d-polytope P^*. If P^* has m facets, then Z_{P^*} is a smooth $(d+m)$-manifold. The interest in them comes from the fact that on the one hand they are defined in combinatorial terms, while on the other hand they are related to the quasi-toric manifolds via a certain universal property.

The last three chapters cover the more recent development with several new results obtained by the authors. For each finite simplicial complex Δ the authors define the moment angle complex Z_Δ, which coincides with Z_{P^*} for $\Delta = \partial P$. In particular, this way one obtains a smooth manifold associated to an arbitrary simplicial sphere. In the construction the authors use cubical complexes in addition to simplicial ones; this is why Chapter 4 contains an introduction to cubical complexes.

The rest of the book investigates the cohomology of Z_Δ. One of the techniques are spectral sequences. Another relies on the following known fact: Simplicial complexes can also be studied via coordinate subspace arrangements. So the results of *M. Goresky* and *R. MacPherson* [‘Stratified Morse theory’, Springer (1988; Zbl 0639.14012)] on the cohomology of the complements of coordinate subspace arrangements can be applied.

On 144 pages the book under review describes the state of the art from the definitions of the most basic notions up to very recent results. The text contains a wealth of material and references to the literature. Proofs are often sketched and sometimes omitted. The authors do give examples, but often only simple ones. Even without considering the new results, the book may be a welcome collection for researchers in the field and a useful overview of the literature for novices.

Reviewer: Michael Joswig (Berlin)

MSC:

52B70	Polyhedral manifolds
14M25	Toric varieties, Newton polyhedra, Okounkov bodies
57Q15	Triangulating manifolds
14F45	Topological properties in algebraic geometry
06A11	Algebraic aspects of posets
52B05	Combinatorial properties of polytopes and polyhedra (number of faces, shortest paths, etc.)
52-02	Research exposition (monographs, survey articles) pertaining to convex and discrete geometry
57-02	Research exposition (monographs, survey articles) pertaining to manifolds and cell complexes
57Q20	Cobordism in PL-topology
14-02	Research exposition (monographs, survey articles) pertaining to algebraic geometry

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moment-angle manifold; quasi-toric manifold; toric variety; convex polytope