Kaczorowski, Jerzy; Perelli, Alberto

On the structure of the Selberg class. I: $0 \leq d \leq 1$. (English) Zbl 1126.11335

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A. Selberg [Proceedings of the Amalfi conference on analytic number theory, held at Maiori, Amalfi, Italy, from 25 to 29 September, 1989. Salerno: Universitá di Salerno, 367–385 (1992; Zbl 0787.11037)] defined a class of Dirichlet series which includes the Riemann zeta-function, Dirichlet L-functions, L-functions attached to modular forms (at least conjecturally), and essentially all Dirichlet series where one might expect a Riemann hypothesis. Namely, functions in this class have precise global analytic behavior, functional equations, Euler products, and tight bounds on the coefficients. A study of this class and Selberg's conjectures has been taken up by other authors (notably Conrey, Ghosh, Bochner, and Murty; references are in the paper under review). A fundamental problem in this arena is the characterization of the admissible functional equations and of the so-called primitive functions in the class.

In this paper, the authors take up some of these questions and provide complete and unconditional results for one of the more interesting subclasses (that containing the Riemann zeta-function, Dirichlet L-functions and Dirichlet polynomials). They determine that these are essentially all the functions in this subclass. This includes and extends results of Conrey-Ghosh and Bochner. The methodology involves some intricate analysis using incomplete Fox hypergeometric functions. But the most interesting aspect of their methodology is the resurrection of Linnik's method of additive twists. The (perhaps a priori more natural) multiplicative twists depend on the Euler product condition of the class definition. This additive approach has the side consequence that the Euler product assumptions are not required; in fact these properties are deduced from the final characterization. This approach also admits a vector-space structure on the subclass; the authors prove dimension results for this space, as well.

Reviewer: James Lee Hafner

MSC:

11M41 Other Dirichlet series and zeta functions
11M06 $\zeta(s)$ and $L(s,\chi)$
11M36 Selberg zeta functions and regularized determinants; applications to spectral theory, Dirichlet series, Eisenstein series, etc. (explicit formulas)

Full Text: DOI

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