Schramm, Oded

Scaling limits of loop-erased random walks and uniform spanning trees. (English)

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The author introduces and analyses the so-called stochastic Löwner evolution (SLE), also named Schramm's process by other authors, which is the conjectured scaling limit for at least two interesting models from statistical mechanics. The existence of a scaling limit for these two models has not been proven yet, but the existence of scaling limits along suited subsequences is easily verified. Under the assumption that the limit actually exists and is conformal invariant, the author identifies them in terms of the SLE and derives some almost sure properties.

The notion of conformal invariance is around in the physicist's literature since a few decades, but has not been specified yet for many important models. The present paper gives a mathematically rigorous sense to the conformal invariance for two interesting models, the loop-erased random walk (LERW) and for the uniform spanning tree (UST). (The author announces to describe also the conjectured scaling limit of critical site percolation by similar means in a forthcoming paper.) Together with recent results on non-intersection exponents for Brownian motions, obtained by the author in collaboration with Lawler and Werner, these are the first examples of this kind and represent a breakthrough in the mathematical understanding of critical phenomena of two-dimensional models from statistical mechanics.

The LERW is a discrete-time random process on $\mathbb{Z}^d$ (here: $d = 2$) evolving in time where any loop that the trajectory closes is immediately removed, such that we obtain a self-avoiding random path. The UST is a random cycle-free connected subgraph of a given finite graph G that contains all the vertices and has the uniform distribution on the set of all such graphs. The notion of a UST may naturally be extended to infinite graphs G, and in the present paper the case of $G = \mathbb{Z}^2$ is considered throughout. There are intimate connections between the LERW and the UST. Two of the main open questions about these two models (and many related ones) are the following. Assume that the above models are defined on the lattice $\delta\mathbb{Z}^2$ rather than on $\mathbb{Z}^2$, is there a natural limiting random process for a properly scaled version of the above process as the mesh $\delta > 0$ tends to zero, and how can this limit be described? The present paper does not answer the first question, but (assuming the answer yes to the first one) the second.

The stochastic Löwner evolution is defined as follows. Let $(B(t))_{t\in[0,\infty)}$ be a standard Brownian motion on the boundary of the unit disc $\mathbb{U} = \{z \in \mathbb{C}\colon |z| < 1\}$, and fix a parameter $\kappa \geq 0$. With $\zeta(t) = B(-\kappa t)$ for $t \leq 0$, solve the so-called Löwner differential equation

$$\frac{\partial f}{\partial t} = z f_t'(z) \frac{\zeta(t) + z}{\zeta(t) - z}, \qquad z \in \mathbb{U}, \quad t \leq 0,$$

with the boundary value $f_0(z) = z$. Then f_t is a conformal mapping from $\mathbb{U}$ into some domain D_t. The process $(\mathbb{U} \setminus D_t)_{t\leq 0}$ is called the SLE. Assuming that the scaling limit of the LERW exists and is conformal invariant, it is proven that, for the choice $\kappa = 2$, this scaling limit has the same distribution as $(f_t(\zeta(t)))_{t\leq 0}$. An analogous assertion is proved for the UST. The choices $\kappa = 6$ and $\kappa = 8$ also lead to interesting processes in terms of which the conjectured scaling limit of critical site percolation and the Peano curve winding around the scaling limit of UST may be described in future work, respectively.

Reviewer: W.König (Berlin)

MSC:

60K35 Interacting random processes; statistical mechanics type models; percolation theory
82B41 Random walks, random surfaces, lattice animals, etc. in equilibrium statistical mechanics
30C35 General theory of conformal mappings

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