Bridson, Martin R.; **Haefliger, André**
Metric spaces of non-positive curvature. (English) Zbl 0988.53001
Grundlehren der Mathematischen Wissenschaften. 319. Berlin: Springer. xxi, 643 p. (1999).

This book is a comprehensive essay on the geometry of complete simply connected metric spaces of non-positive curvature in the sense of A. D. Alexandrov, and of the isometry groups of such spaces. Non-positive curvature is expressed here in terms of the CAT(0) comparison inequality, of which we recall the definition. The metric spaces under consideration are *geodesic*, that is, each pair of points in such a space can be joined by a geodesic arc which, by definition, is an arc whose length is equal to the distance between its two endpoints. A geodesic space X satisfies the CAT(0) property if the following holds : Let x, y and z be three arbitrary points in X, and let $[x,y]$, $[y,z]$ and $[z,x]$ be three geodesic arcs joining them. Let Δ be a triangle in the Euclidean plane with vertices x', y' and z', and sides $[x',y']$, $[y',z']$ and $[z',x']$ whose lengths are equal respectively to those of $[x,y]$, $[y,z]$ and $[z,x]$. Then, the natural map from the union $[x,y]\cup[y,z]\cup[z,x]$ (with distances between points measured in X) to the edges of Δ is distance non-decreasing. The triangle Δ is said to be a *comparison triangle* for the "triangle" $[x,y]\cup[y,z]\cup[z,x]$ in X.

The interest in CAT(0) spaces, and more generally in CAT(κ) spaces, with κ being any real number has been revived in the last 20 years by M. Gromov, who introduced the terminology CAT(κ) in honor of E. Cartan, A. D. Alexandrov and A. Toponogov. (The definition of a CAT(κ) space is obtained by modifying that of a CAT(0) space by taking the comparison triangle Δ to be in the complete simply-connected metric space of constant curvature κ instead of in the Euclidean plane.)

The book under review offers a beautiful and most complete introduction to the basic properties spaces of CAT(κ) spaces, as well as an exposition of the most important results in the subject.

The book is divided into three parts. Part I is an introduction to the geometry of geodesic metric spaces, Part II contains the basics of CAT(κ) spaces, and Part III contains specialized results on the subject. Let us describe briefly the content of each part.

Part I of this book constitutes a basic text on geodesic metric spaces. The topics which are discussed start with the definition of a metric space, and they include the study of induced metrics on covering spaces of a metric space, Hilbert spaces, ℓ^p spaces, basic constructions like cones, joins, limits, ultralimits and asymptotic cones, metrics on polyhedral complexes (in particular cubical complexes), the study of links and geodesics in such metric complexes, and a general study of geodesic actions on metric spaces: ends, growth, rigidity, quasi-isometric invariants, approximations by metric graphs, and several other topics.

In Part II, the authors give several equivalent formulations of the CAT(κ) condition. Some of these formulations involve angles in metric spaces (a notion which has also been worked out by Alexandrov), other conditions are based on convexity properties of the distance function (work of Alexandrov, and also, independently of H. Busemann), or orthogonal projections, and so on.

The important results of Part II include those which describe global properties of the space, derived from local ones. For instance, the authors give a proof of a generalized version due to Gromov (with variations of proofs due to W. Ballman and to S. Alexander and C. Bishop) of a theorem of Cartan and Hadamard in the case of manifolds of non-positive curvature. One version of the general theorem says that if X is a complete metric space whose distance function satisfies a local convexity condition, then the distance function of the induced metric on the universal covering of X satisfies a global convexity property. Another important result proved in the same context of local-implies-global is a result establishing a necessary and sufficient condition (also due to Gromov) on the existence of CAT(0) metrics on polyhedral complexes in terms of properties of the links of the vertices. These conditions give practical methods for deciding if a given complex carries a metric of non-positive curvature. The authors consider interseting examples arising from geometric group theory and involving cubical complexes, two-dimensional complexes, knot and link groups, and they discuss related algorithmic problems.

Part II also contains a study of groups acting by isometries on CAT(0) spaces. After establishing the basic properties of individual isometries and their classification in terms of the properties of their displacement functions, the authors show that the group of isometries of a compact non-positively curved space is a topological group with finitely many connected components, the component of the identity being a torus. They give then generalized versions of structure theorems due to Gromoll-Wolf and Lawson-Yau for fundamental groups of compact non-positively curved Riemannian manifolds. The generalized versions apply to groups acting properly and cocompactly by isometries on CAT(0) spaces.

Part II contains then a study of the boundary at infinity of a CAT(0) space X. The authors describe the boundary ∂X as a set of equivalence classes of geodesic rays, and a natural topology on $\overline{X} = X \cup \partial X$ which is called the cone topology. They give other descriptions of $\overline{X}$ by considering the closure of X in the Banach space of continuous functions on X modulo additive constants (X being embedded *via* the distance functions). Here, the points in $\overline{X}$ emerge as Busemann functions. The authors describe a useful class of metrics on ∂X, using the notion of angle between geodesic rays starting at some fixed point in X. The associated topology is in general weaker than the cone topology. If X is a complete CAT(0) space, then ∂X equipped with this metric is a CAT(1) space. There is an induced *length metric* on ∂X which is called the *Tits metric* and which encodes the structure of flat subspaces in X.

Part II contains several sections describing examples of CAT(0) spaces. In particular, the authors develop gluing techniques which allow one to construct new CAT(0) spaces out of basic ones. Other examples include spaces arising from the theory of symmetric spaces, and Part II of the book includes a large section on these spaces.

In the last section in Part II, the authors introduce the theory of simple complexes of groups which is related to a discussion in Part III on the theory of complexes of groups.

One of the topics discussed in Part III is that of visibility spaces. The visibility property is a generalization of the visibility condition on smooth manifolds which had been introduced in a paper by P. Eberlein and B. O'Neill in 1973, and it expresses the fact that the space is "negatively curved on the large scale". The authors prove that if X is a CAT(0) space which is proper (meaning that every closed ball is compact) and which admits a cocompact group of isometries, then X is a visibility space if and only if it does not contain an isometrically embedded Euclidean plane.

The authors then lay out the foundations of the theory of complexes of groups and, in particular, of non-positively curved complexes of groups. Complexes of groups, which were introduced by Haefliger, are generalizations of graphs of groups, which were invented by H. Bass and J. P. Serre. In this book, the authors use this theory to discuss the *developability theory* for non-positively curved complexes of groups. Here, a complex of groups is developable if it arises from a certain action involving combinatorial descriptions of polyhedral complexes. The authors prove a general theorem which in a particular case gives a result of Gromov stating that every complete Riemannian orbifold of non-positive curvature is developable. The book contains also basic material such as a detailed study of fundamental groups and coverings of a complex of groups, groupoids of local isometries, orbifolds, étale groupoids, fundamental groups and coverings of étale groupoids.

Part III contains then a section on Gromov δ-hyperbolicity, with its relation to CAT(0) theory and to area and isoperimetric inequalities. They prove such important (and by now classical) results as subquadratic isoperimetric inequality implies a linear one.

Part III contains again profound results on groups acting properly and cocompactly by isometries on CAT(0) spaces. The authors analyse algorithmic properties of such groups, and they study their subgroups. They show that many theorems concerning groups of isometries of CAT(0) spaces can be extended to hyperbolic and semihyperbolic groups.

In conclusion, it can be said that the book is an indispensable reference and a very needful tool for graduate students who want to learn this theory as well as for researchers working in the subject. The exposition is clear, the proofs are complete, and some of the advanced results that are discussed are original. Every section of the book contains interesting historical remarks and comments.

Reviewer: Athanase Papadopoulos (Strasbourg)

MSC:

53-02 Research exposition (monographs, survey articles) pertaining to differential geometry
53C23 Global geometric and topological methods (à la Gromov); differential geometric analysis on metric spaces
53C70 Direct methods (G-spaces of Busemann, etc.)
53C45 Global surface theory (convex surfaces à la A. D. Aleksandrov)
20F65 Geometric group theory
57M07 Topological methods in group theory

Keywords:

non-positive curvature; Gromov hyperbolic; comparison inequality; geodesic; metric space; geometric group theory; quasi-isometry; complex of groups; polyhedral complex; visibility; boundary; cone topology; CAT(0) space; étale groupoid; group action