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Projective elements in K-theory and self maps of ΣCP^∞. (English) Zbl 0924.55002
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The author works in the homotopy category of based spaces and based maps. Given a space X, let the reduced K-theory be denoted by $K(X)$ and the homology group of integral coefficients by $H_*(X)$. Let $\mathbb{C}\mathrm{P}^\infty$ be the infinite-dimensional complex projective space. Let η be the canonical line bundle over $\mathbb{C}\mathrm{P}^\infty$ and $i:\mathbb{C}\mathrm{P}^\infty \to \mathrm{BU}$ be the classifying map of the virtual bundle $\eta-1$. Since BU has a loop space structure, there exists a unique extension of i to the loop map $j:\Omega\Sigma\mathbb{C}\mathrm{P}^\infty \to \mathrm{BU}$.

In this paper, the author investigates the following problems: Given an element $\alpha \in K(X)$, when does there exist a lift $\widehat{\alpha} \in [X, \Omega\Sigma\mathbb{C}\mathrm{P}^\infty]$ such that $j_*(\widehat{\alpha}) = \alpha$? If α has a lift, how can we construct the lift $\widehat{\alpha}$? Define $\mathrm{PK}(X) = \{\alpha \in K(X) \mid \exists\widehat{\alpha} \in [X, \Omega\Sigma\mathbb{C}\mathrm{P}^\infty] \text{ such that } j_*(\widehat{\alpha}) = \alpha\}$. If an element $\alpha \in K(X)$ belongs to $\mathrm{PK}(X)$, one says that α is projective. The significance of the above problem is as follows: The James splitting theorem [*I. M. James*, "The topology of Stiefel manifolds", Lect. Note Series 24 (1976; Zbl 0337.55017)] implies that there exists a loop map $\theta:\mathrm{BU} \to \Omega^\infty\Sigma^\infty\mathbb{C}\mathrm{P}^\infty$ such that $\theta \circ j = E^\infty : \Omega\Sigma\mathbb{C}\mathrm{P}^\infty \to \Omega^\infty\Sigma^\infty\mathbb{C}\mathrm{P}^\infty$. Therefore, given an element $\alpha \in K(X)$, the stable map, ad $j\circ(\theta(\alpha)) : \Sigma^\infty X \to \Sigma^\infty\mathbb{C}\mathrm{P}^\infty$ can be considered. Using the information of $K(X)$, the induced homomorphism [*C. A. McGibbon*, Trans. Am. Math. Soc. 271, 325-346 (1982; Zbl 0491.55014); *K. Morisugi*, Publ. Res. Inst. Math. Sci. 24, No. 2, 301-309 (1988; Zbl 0657.55010)] of ad $j\circ(\theta(\alpha))_* : H_*(X) \to H_*(\mathbb{C}\mathrm{P}^\infty)$ can be calculated. If α has a lift $\widehat{\alpha}$, then this implies that the stable map ad $j\circ(\theta(\alpha))$ and its induced homomorphism come from the unstable map ad $j\circ(\widehat{\alpha}) : \Sigma X \to \Sigma\mathbb{C}\mathrm{P}^\infty$. These imply that the determination of $\mathrm{PK}(X)$ gives complete information of the image of the homomorphism: $[\Sigma X, \Sigma\mathbb{C}\mathrm{P}^\infty] \to \mathrm{Hom}(H_*(X), H_*(\mathbb{C}\mathrm{P}^\infty))$. However, since the above factors through $\mathrm{Hom}(H_*(X), H_*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty))$, it is desirable to obtain the image of $[X, \Omega\Sigma\mathbb{C}\mathrm{P}^\infty] \to \mathrm{Hom}(H_*(X), H_*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty))$. So, if possible, it is preferable to have the information not of ad $j\circ(\widehat{\alpha})_*$ but $\widehat{\alpha}_* : H_*(X) \to H_*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty)$. For this the geometry of the lift $\widehat{\alpha}$ is necessary.

In this context, the author proves five theorems and a corollary. In Theorem 1.1 it is proved that if $\mathbb{C}\mathrm{P}^\infty \wedge \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ is the adjoint of the Hopf construction for the H-space $\mathbb{C}\mathrm{P}^\infty$, then this map has an extension $\# : \Omega\Sigma\mathbb{C}\mathrm{P}^\infty \wedge \Omega\Sigma\mathbb{C}\mathrm{P}^\infty$ such that $j\circ\# = \bigotimes\circ(j\wedge j)$, where $\bigotimes : \mathrm{BU}\wedge\mathrm{BU} \to \mathrm{BU}$ is the map which represents the external tensor product $K(X)\otimes K(Y) \to K(X\wedge Y)$. In Theorem 1.2 the properties of $\mathrm{PK}(X)$ are established. Theorem 1.4 contains an evaluation of the commutator in the group $[\mathbb{C}\mathrm{P}^\infty, \Omega\Sigma\mathbb{C}\mathrm{P}^\infty]$ and in the Theorem 1.5 the Hurewicz homomorphism $h : \pi_*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty) \to H_*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty)$ is studied. As a corollary of Theorem 1.5 it is proved that the group $[\Sigma\mathbb{C}\mathrm{P}^n, \Sigma\mathbb{C}\mathrm{P}^n]$ is not commutative for $n \geq 3$. Theorem 1.7 is a technical result concerning the composition structure of the adjoints of some maps $f_n : \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ inductively defined starting from the inclusion $f_1 : \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$.

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