zbMATH Open
THE FIRST RESOURCE FOR MATHEMATICS

Cox, David A.; **Katz, Sheldon**

Mirror symmetry and algebraic geometry. (English) Zbl 0951.14026

Mathematical Surveys and Monographs. 68. Providence, RI: American Mathematical Society (AMS). xxi, 469 p. (1999).

Mirror symmetry is a fascinating, still somewhat mysterious phenomenon in complex geometry, which has its intuitive origin in the physics of quantum fields. About ten years ago, this phenomenon has exploded onto the mathematical scene, metaphorically speaking, and grown into one of the central fields of current mathematical research ever since. In two-dimensional superconformal quantum field theory, various attempts have been made to construct phenomenologically realistic superstring models. Most approaches to such a theory assume a vacuum configuration that contains a compact complex Calabi-Yau threefold as a Cartesian factor. Then, in those approaches, mirror symmetry is the physically compelling stipulation that the possible Calabi-Yau models should appear in pairs, namely as so-called "mirror pairs". In the sense of this non-mathematical definition from physics, two Calabi-Yau models are said to form a mirror pair if their sigma models induce isomorphic superconformal field theories whose ($N = 2$)-superconformal representations coincide up to the sign of a certain quantum number. Based on their understanding of evidence, physicists are quite confident about the mathematical validity of mirror symmetry. In fact, both their specific kind of intuition and their rich experience with mathematical manipulations which have not yet been mathematically justified, in the rigorous sense of mathematics, have led physicists to some astonishing predictions in complex geometry. As to the phenomenon of mirror symmetry in Calabi-Yau geometry, the story probably started with the 1991 paper by *P. Candelas*, *X. de la Ossa*, *P. S. Green* and *L. Parkes* [in: Essays on Mirror Manifolds, 31-95 (1992; Zbl 0826.32016)] in which there are presented some predictions about the enumerative geometry of rational curves on the quintic threefold. At that time, these predictions went far beyond anything algebraic geometry could handle by means of its (already monstrous) toolkit. Thus, once more in mathematical history, the challenge for mathematicians was to grasp a new phenomenon emerged from mathematical physics and, even more importantly, to prove some of the predictions made by the physicists.

Indeed, over the past ten years, complex geometers have made substantial progress in creating a rigorous mathematical foundation for aspects of mirror symmetry, and this utmost intensive process has even procreated new fields of research in complex geometry, above all in algebraic geometry. – The most spectacular examples, in this regard, include quantum cohomology, the theory of Gromov-Witten invariants, Kontsevich's definition of stable maps, the moduli theory for Calabi-Yau manifolds, Batyrev's duality theory for certain toric varieties, Givental's concept of quantum differential equations, and other inventive approaches to the phenomenon of mirror symmetry. The rapidly increasing number of new concepts, methods, techniques, fundamental results, discovered interrelations, and related high-quality research papers has become nearly overwhelming, in the meantime, and makes it difficult for non-specialists to keep up with these galloping developments in both mathematics and impelling quantum physics.

The book under review presents the very first comprehensive introduction to the algebro-geometric aspects of mirror symmetry.

Taking into account that a reader who wants to learn about mirror symmetry has to face many tough obstacles, the authors have tried to arrange the material, in its full variety and complexity, in such a way that the entire text appears as being largely self-contained, systematizing, throughout well-motivated, reasonably detailed and complete, comprehensible, inspiring, and complementary to the current research literature. The book was primarily written to address the recent mathematical framework of mirror symmetry and, as far as possible, to show how this mathematical abstraction reflects the original spirit of the underlying physics. Accordingly, the authors had two primary target audiences in mind: Mathematicians wanting to learn about the recent developments in mirror symmetry, and physicists who are acquainted with mirror symmetry wanting to learn about the solid mathematical background of the subject. With all these ambitious, high-minded and methodologically very challenging goals in mind, the authors have succeeded to write a masterpiece of an encyclopaedic textbook that really leaves nothing to be desired.

The contents of the book are arranged as follows: In their preface to the book, the authors explain the main goals they are striving for, the relation between mathematics and physics reflected by the text, and

© 2021 FIZ Karlsruhe GmbH

how to read the book in different ways. Then the text is divided into twelve chapters.

Chapter 1: "Introduction": This chapter starts with a discussion of the physics that led to the idea of mirror symmetry in the category of Calabi-Yau manifolds, including correlation functions in superconformal quantum field theories and Yukawa couplings, and ends with a preview of the algebraic geometry to be explored in the remaining eleven chapters of the book.

Chapter 2: "The quintic threefold": This chapter is devoted to the geometry of the quintic threefold, the first example for which mirror symmetry was used to make enumerative predictions on curves in Calabi-Yau manifolds. The authors review the original approach by Candelas-de la Ossa-Green-Parkes (1991) and, thereafter, the alternative enumeration methods developed by D. Morrison (1993), A. Givental (1996), and B. Lian-K. Liu-S.-T. Yau (1997).

Chapter 3: "Toric geometry": This chapter provides the basic mathematical background material from the theory of toric varieties over the field of complex numbers, as far as it is needed in the sequel. – In the second part of this chapter, the authors focus on those aspects of toric geometry that are most relevant to mirror symmetry. This includes Kähler cones and Mori cones of simplicial toric varieties, symplectic methods to construct toric varieties, the GKZ decomposition, Fano properties of toric varieties, automorphisms of toric varieties, and three typical examples. Much of the material is fairly recent and compiled from the deep work of V. Batyrev, D. Cox, M. Demazure, T. Oda, and others.

Chapter 4: "Mirror symmetry constructions": This chapter describes the various available constructions of mirror manifolds for Calabi-Yau threefolds, with a special emphasis on the approaches by *V. Batyrev* (via reflexive polytopes in toric geometry), *V. Batyrev* and *L. Borisov* (via nef-partitions and reflexive Gorenstein cones in rational polyhedral cones), *C. Voisin* and *C. Borcea* (independently via Nikulin's families of $K3$ surfaces with involution), and their interrelations. The "classical" case of the quintic threefold is revisited in the frame of Batyrev mirrors.

Chapter 5: "Hodge theory and Yukawa couplings": In this chapter, the authors recall and develop some basic background material from the Hodge theory of projective complex manifolds, and apply this to the calculation of the Yukawa couplings of a Calabi-Yau manifold. This incorporates variations of Hodge structures, the geometry of the Picard-Fuchs equations, the maximally unipotent monodromy and the concept of moduli of Calabi-Yau manifolds, the Griffiths-Dwork method for calculating the Picard-Fuchs ideal for projective, weighted projective, or toric hypersurfaces. Then, after a discussion of the hypergeometric systems studied by *I. M. Gelfand*, *M. M. Kapranov* and *A. V. Zelevinskij* [Adv. Math., 84, No. 2, 255-271 (1990; Zbl 0741.33011)], which provides an alternate approach to computing Picard-Fuchs equations, the authors turn to the Yukawa couplings of Calabi-Yau threefolds. In the course of this discussion, which is essential for constructing mirror maps later on, a simplified proof of Deligne's theorem on the relation between normalized Yukawa couplings and the Gauss-Manin connection on Calabi-Yau threefolds is given in a very elegant way.

Chapter 6: "Moduli spaces": The goal of this chapter is to clarify, mathematically, what is meant by the so-called "mirror map" of Calabi-Yau manifolds. To this end, a full understanding of both the complex moduli space and the Kähler moduli space of a Calabi-Yau manifold is required, and the construction of these objects occupies the first part of this chapter. The authors devote special attention to Calabi-Yau hypersurfaces in toric varieties, for in these cases at least the polynomial and toric parts of these moduli spaces can be described explicitly. The second part of this chapter deals then with the construction of the mirror map between the complex moduli space of a Calabi-Yau manifold and the Kähler moduli space of its mirror manifold as constructed in chapter 4. This mirror map appears as a local isomorphism at distinguished points in the corresponding moduli spaces, and its description uses the full mathematical machinery developed in chapters 3, 4 and 5.

Chapter 7: "Gromov-Witten invariants". The objects of study in this chapter are the different kinds of Gromov-Witten invariants, which are, in either setting, a crucial ingredient for the rigorous mathematical definition of A-model Yukawa couplings and correlation functions. The authors emphasize the algebraic approaches to the construction of Gromov-Witten invariants using moduli of stable maps (à la *M. Kontsevich, K. Behrend* and *Yu. Manin, W. Fulton* and *R. Pandharipande*, and others) and virtual fundamental classes (à la *K. Behrend* and *B. Fantechi, J. Li* and *G. Tian, E. Getzler*, and others), but they also discuss the approaches via symplectic geometry and pseudo-holomorphic curves due to *Y. Ruan, G. Tian, B. Siebert*, and others. The basic properties of the Gromov-Witten classes, i.e., their stipulated axioms, are explained in a very thorough manner, and their enumerative significance as well as their computational treatment are illustrated by means of various concrete examples. Also, the authors have included a nice application yielding a subtle relation between the number of degree-10 rational curves on the "classical" quintic threefold and a certain instanton number.

Chapter 8: "Quantum cohomology": Quantum cohomology is the final ingredient needed in order to understand the mathematical version of mirror symmetry. In this chapter, the authors introduce the two basic types of quantum cohomology, namely the so-called big and small quantum cohomology rings of a smooth projective variety.

Using the formalism of algebraic Gromov-Witten invariants, the Gromov-Witten potential (in genus zero) satisfying the celebrated Witten-Dijkgraaf-Verlinde-Verlinde equation (WDVV), and some elements of the Dubrovin-Kontsevich-Manin theory of Frobenius (super-)manifolds [cf. *Yu. I. Manin*, "Frobenius manifolds, quantum cohomology, and Moduli spaces", Colloquium Publ. 47 (1999)], the authors describe how a natural variation of Hodge structure on the Kähler moduli space of a Calabi-Yau manifold can be obtained, and how various mathematically precise, Hodge-theoretic versions of the mirror conjecture can be formulated, at least so for (toric) threefolds.

Chapter 9: "Localization": This and the next two chapters are dedicated to developing some more advanced mathematical machinery needed for a deeper understanding of the mathematical version of mirror symmetry, and to proving some instances of this phenomenon. For this purpose, chapter 9 first reviews the basics from equivariant cohomology theory, together with the corresponding localization principle due to Atiyah-Bott (1984), and turns then to Kontsevich's approach (1994) to counting degree-d rational curves on the quintic threefold by applying the localization principle to certain moduli spaces of stable maps. This chapter ends with a brief discussion of some examples for equivariant Gromov-Witten classes.

Chapter 10: "Quantum differential equations": This chapter mainly explains *A. Givental*'s special approach to the mathematical treatment of mirror symmetry. Starting from a mathematical description of the so-called gravitational correlators, which are generalizations of the algebraic Gromov-Witten invariants, the authors describe Givental's twisted version of the Dubrevin connection on the big quantum cohomology ring of a projective manifold, its flat sections, and the relations to quantum cohomology, in general, via Givental's J-function. Quantum differential operators, which are defined to be differential operators annihilating Givental' s J-function, play an essential role in this context, and lead to fundamental links between Picard-Fuchs equations and certain identities in quantum cohomology rings. Several concrete examples, including the physically relevant Calabi-Yau threefolds, illustrate the significance of Givental's approach to investigating Gromov-Witten potentials in quantum cohomology. This chapter also contains a brief discussion of the intriguing "Virasoro conjecture" concerning gravitational correlators in quantum cohomology and Virasoro algebras.

Chapter 11: "The mirror theorem": This chapter represents the highlight of the entire text. Combining all the mathematical concepts, methods, techniques and results developed in the previous chapters, the authors describe some of the recently accomplished proofs of the existence of a mathematical mirror symmetry in special cases of projective Calabi-Yau manifolds. In addition to the Hodge-theoretic version of the mirror theorem stated in chapter 8, they focus here on two closely related approaches to the mirror conjecture, both of which are based on equivariant intersection theory in the moduli spaces of stable maps. At first, the proof of the mirror theorem for the (physically important) quintic threefold, due to Lian-Liu-Yau [cf. *B. Lian*, *K. Liu* and *S.-T. Yau*, "Mirror principle. I, "Asian J. Math. 1, No. 4, 729-763 (1997)], is thoroughly discussed. The second part of this chapter is dedicated to Givental's variant of the mirror theorem for nef complete intersections in projective spaces and its extension to toric complete intersections, homogeneous spaces, and special higher-dimensional Calabi-Yau manifolds [cf. *A. Givental*, Int. Math. Res. Not. 1996, No. 13, 613-663 (1996; Zbl 0881.55006)].

Chapter 12: "Conclusion": This concluding chapter gives a brief summary of what has been achieved in the course of the entire text, and of what remains to be done in mirror geometry. The authors bring together most of the open problems and conjectures mentioned in earlier chapters and discuss briefly some of the tremendously many other aspects of mirror symmetry that could not be treated in the book. For the sake of self-containedness of the text as well as for the convenience of the reader, the authors have added two appendices to the contents.

Appendix A: "Singular varieties": As singular algebraic varieties occur in some places of the text, especially in chapter 4, this appendix compiles some facts on canonical and terminal singularieties (à la M. Reid), Cohen-Macaulay varieties, and the algebraic geometry of orbifolds.

Appendix B: "Physical theories": This appendix summarizes some basic key points of physical theories mentioned in the book. The authors explain the basics from both the Lagrangian and the Hamiltonian formulation of general quantum field theories, nonlinear sigma models, conformal field theories, Landau-

© 2021 FIZ Karlsruhe GmbH

Ginzburg models, and gauged linear sigma models (after E. Witten). The final section of this appendix provides a brief introduction to topological quantum field theories in their axiomatic setting, together with some hints to their connection with Frobenius algebras and quantum cohomology.

Although the number of papers on mirror symmetry and its related topics is unbelievably large, and rapidly increasing, the authors have managed to make the bibliography fairly up-to-date and complete. The bibliography lists up more than 300 references, most of which have been published within the last three years, and all these quotations are commented on somewhere in the text. In view of this huge amount of very recent research material which the authors have systematically processed in their book, and with regard to the very fact that a nearly crushing abundance of new and highly advanced, perhaps even fancy mathematics had to be worked up, it is obvious that many of the highly non-trivial proofs explained in the text are not complete. In those cases, the authors give guiding hints to the original papers, and point out what strategy for further reading would be best.

Altogether, the authors have written a brilliant, almost encyclopaedic introduction to the brand-new field of mathematical mirror symmetry in its algebro-geometric setting. The numerous illustrating worked examples as well as the inclusion of the relevant physical background material, apart from the ubiquitous abstract depth and abundant variety of mathematical methods characterizing this text, make this book particularly valuable and somehow unique. Together with the recent monographs "Mirror symmetry" by *C. Voisin* [cf. SMF/AMS Texts Monogr. 1 (1999; Zbl 0945.14021)] and "Frobenius manifolds, quantum cohomology, and moduli spaces" by *Yu. I. Manin* (log.) this book provides a standard text of the subject of quantum cohomology and mirror symmetry.

Reviewer: W.Kleinert (Berlin)

MSC:

14J32	Calabi-Yau manifolds (algebro-geometric aspects)
14-02	Research exposition (monographs, survey articles) pertaining to algebraic geometry
81-02	Research exposition (monographs, survey articles) pertaining to quantum theory
14N35	Gromov-Witten invariants, quantum cohomology, Gopakumar-Vafa invariants, Donaldson-Thomas invariants (algebro-geometric aspects)
81T40	Two-dimensional field theories, conformal field theories, etc. in quantum mechanics
81T45	Topological field theories in quantum mechanics
32Q25	Calabi-Yau theory (complex-analytic aspects)
81T60	Supersymmetric field theories in quantum mechanics
14M25	Toric varieties, Newton polyhedra, Okounkov bodies
14D21	Applications of vector bundles and moduli spaces in mathematical physics (twistor theory, instantons, quantum field theory)
14J80	Topology of surfaces (Donaldson polynomials, Seiberg-Witten invariants)

Cited in **6** Reviews
Cited in **219** Documents

Keywords:

mirror pairs; mirror symmetry; Calabi-Yau models; enumerative geometry of rational curves; quintic threefold; Gromov-Witten invariants; stable maps; duality theory for toric varieties; quantum differential equations; Kähler cones; Mori cones; variations of Hodge structures; Picard-Fuchs equations; unipotent monodromy; Gauss-Manin connection; equivariant cohomology; terminal singularieties

© 2021 FIZ Karlsruhe GmbH