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Wavelets, paraproducts and Navier-Stokes. With a preface by Yves Meyer. (Ondelettes, paraproduits, et Navier-Stokes.) (French) Zbl 1049.35517
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From the introduction: An approximation of the point of view that we adopt here is provided by the famous allegory of energy cascades proposed in 1922 by the English meteorologist *L. F. Richardson* [Weather prediction by numerical process, Cambridge Univ. Press, Cambridge (1922; JFM 48.0629.07)]. Here is what it involves: One seeks to decouple the Navier-Stokes equations into (a) a sequence E_j ($j \in \mathbb{Z}$) of equations describing the phenomena that develop at scale 2^{-j} (decoupled equations) and (b) a family F_j ($j \in \mathbb{Z}$) of transfer equations describing how the phenomena at scale 2^{-j} influence the phenomena at scale 2^{-j-1} (twice smaller).

Numerous researchers have been interested in finding a hierarchical law in the transfer of turbulent energy to the small scales. The most famous of this research involving the notion of energy cascades is the theory of fully developed turbulence proposed in 1941 by A. N. Kolmogorov.

Here, following the approach of *P. Frick* and *V. D. Zimin* [see, e.g., in Wavelets, fractals, and Fourier transforms, Oxford: Clarendon Press. Inst. Math. Appl. Conf. Ser., New Ser. 43, 265–283 (1993; Zbl 0821.76036)], we seek to carry out, simultaneously, a scale analysis and a frequency analysis (with scale 2^{-j} associated with frequency 2^j). The principal goal of our work is to prove, using, like Frick and Zimin, the Littlewood-Paley decomposition (Chapter I), or, like *P. Federbush* [Commun. Math. Phys. 155, No. 2, 219–248 (1993; Zbl 0795.35080)], wavelet decomposition (Chapter II), the well-foundedness of the heuristics introduced by Richardson. Moreover, these decompositions (Littlewood-Paley or wavelet) are naturally adaptable to the study of the bilinear term (*J.-M. Bony*'s paraproduct algorithms [Ann. Sci. Éc. Norm. Supér. (4) 14, No. 2, 209–246 (1981; Zbl 0495.35024)]).

More precisely, following Federbush's original approach [op. cit.], we seek to derive a solution of the Navier-Stokes equations in dimension three by writing (2) $\mathbf{v}(t,\mathbf{x}) = S(t)\mathbf{v}_0 + \mathbf{w}(t,\mathbf{x})$, where $\mathbf{v}_0(\mathbf{x})$ is the initial condition and $S(t) = \exp(t\Delta)$ the heat semigroup, $S(t)\mathbf{v}_0$ is the 'tendency' of $\mathbf{v}(t,\mathbf{x})$ (solution of the heat equation with the same initial condition $\mathbf{v}_0$), and $\mathbf{w}(t,\mathbf{x})$ is the 'fluctuation' of $\mathbf{v}(t,\mathbf{x})$ (around the tendency). This means that $\mathbf{w}(t,\mathbf{x})$ is oscillating and thus has zero integral (or zero mean value, depending on the case) and that $\mathbf{w}(t,\mathbf{x})$ is an error term compared with the principal term. Moreover, $\mathbf{w}(t,\mathbf{x})$ can be represented as a superposition (linear combination) of oscillating elementary contributions $\mathbf{w}(s,t,\mathbf{x})$, where $0 < s < t$ and the frequencies of these oscillations are (approximately) $1/\sqrt{t-s}$, $0 < s < t$ (Proposition 1.2.9).

The nonlinear mechanism (which is described by Richardson's allegory) is analyzed inside each elementary contribution $\mathbf{w}(s,t,\mathbf{x})$. This contribution is obtained by applying a 'band-pass filter' (coordinated with the frequency $1/\sqrt{t-s}$) to the products of the components of $\mathbf{v}(s,\mathbf{x})$. Moreover, the heat semigroup $S(t)$ plays the role of a 'low-pass filter' associated with the cutoff frequency $1/\sqrt{t}$.

Our goal is specifically to justify the decomposition (2) into tendency and fluctuation by using various functional norms $\|\cdot\|_E$. This set of problems leads us first of all to look for $\mathbf{v}(t,\mathbf{x})$ in the space of continuous functions in t with values in a Banach space E and to write the Navier-Stokes equations in the mild integral form given by the variation of constants formula, i.e., (3) $\mathbf{v}(t) = S(t)\mathbf{v}_0 + \mathbf{B}(\mathbf{v},\mathbf{v})(t)$, where

$$\mathbf{B}(\mathbf{v},\mathbf{u})(t) = -\int_0^t \mathbf{P}S(t-s)\nabla\cdot(\mathbf{v}\otimes\mathbf{u})(s)ds, \tag{4}$$

with $\mathbf{P}$ the projection operator onto the divergence-free vector fields.

We propose to give a precise estimate of the difference $\|\mathbf{v}(t) - S(t)\mathbf{v}_0\|_E = \|\mathbf{B}(\mathbf{v},\mathbf{v})(t)\|_E$ between the nonlinear evolution $\mathbf{v}(t)$ governed by the Navier-Stokes equations and the linear evolution in which we solve the heat equation with the same initial condition. In principle, this question is pertinent in two distinct contexts: (i) the Navier-Stokes equations are solved by a standard algorithm and one seeks to learn a little more about that solution, and (ii) (this being the approach adopted here) one 'does everything at once' and tries to construct $\mathbf{v}(t)$ using an algorithm that takes into account the a priori properties of the E-norm. That these two points of view are not necessarily equivalent is due to the fact that there is no general uniqueness theorem in dimension three.

More precisely, we propose to study Banach spaces E such that, for all $\mathbf{v}_0 \in E$ satisfying $\nabla \cdot \mathbf{v}_0 = 0$ in the sense of distributions, there exist $T > 0$ and a mild solution $\mathbf{v}(t, \mathbf{x})$ of the Navier-Stokes equations (local if $T < \infty$, global if $T = \infty$) such that $\mathbf{v}(t, \mathbf{x}) \in \mathcal{C}([0, T); E)$ and $\mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x})$.

Schematizing, one can say that such a problem consists of writing the Navier-Stokes equations in the form (3) and of establishing, a priori, the continuity of the bilinear operator $\mathbf{B}(\mathbf{v}, \mathbf{u})$ in the space $X = \mathcal{C}([0, T); E)$, i.e., (5) $\|\mathbf{B}(\mathbf{v}, \mathbf{u})\|_N \leq \eta(T)\|\mathbf{v}\|_N\|\mathbf{u}\|_N$, where (6) $\|\mathbf{v}\|_N = \sup_{0\leq t\leq T} \|\mathbf{v}(t, \cdot)\|_E$ is the natural norm in $\mathcal{C}([0, T); E)$.

Once we establish estimate (5), we obtain, using the Picard contraction algorithm (Lemma 1.2.6), the existence of a solution $\mathbf{v}(t, \mathbf{x}) \in \mathcal{C}([0, T); E)$ under the condition (7) $4\eta(T)\|S(t)\mathbf{v}_0\|_N < 1$.

In this text, we examine four methods in succession: 1. In the first two chapters we use the Littlewood-Paley decomposition (Chapter I) and wavelet decomposition (Chapter II) to give sufficient conditions on the Banach space E such that the fundamental estimate (5) holds. Thus we obtain the notion of an adapted space with respect to the Navier-Stokes equations (Definition 1.2.2) for which (5) holds with a continuity constant $\eta(T)$ such that (8) $\lim_{T\to 0} \eta(T) = 0$. Moreover, the norm in these spaces has the property (9) $\|S(t)\mathbf{v}_0\|_N = \|\mathbf{v}_0\|_E$. Thanks to the a priori estimates (5)–(9), we derive, for all $\mathbf{v}_0$ in an adapted space E which satisfy $\nabla \cdot \mathbf{v}_0 = 0$, the existence of a $T = T(\|\mathbf{v}_0\|) > 0$ and the existence and uniqueness of a local solution $\mathbf{v}(t, \mathbf{x})$ of the Navier-Stokes equations in $\mathcal{C}([0, T); E)$ (Theorem 1.2.3). This general theorem allows us to recover some classical results in the Lebesgue spaces $L^p(\mathbb{R}^3)$, $p > 3$, the Sobolev spaces $H^s(\mathbb{R}^3)$, $s > \frac{1}{2}$, and the Morrey-Campanato spaces $M_2^p(\mathbb{R}^3)$, $p > 3$, but it is also applied to the Sobolev-Bessel spaces $L^{p,s}(\mathbb{R}^3)$, the Hölder-Zygmund spaces $C^\alpha(\mathbb{R}^3)$, and, in general, the Besov spaces $B_p^{\alpha,q}(\mathbb{R}^3)$ and the Triebel-Lizorkin spaces $F_p^{\alpha,q}(\mathbb{R}^3)$ (Section 1.3).

2. We show by means of a counterexample that for (5)–(9) to hold it is not necessary that E be adapted (Section 1.2.4). We again obtain a local existence and uniqueness theorem for solutions of the Navier-Stokes equations in $\mathcal{C}([0, T); E)$, with E now a nonadapted space (Theorem 1.2.10).

3. Incidentally, there are many examples of Banach spaces for which it is possible to solve the Navier-Stokes equations even if the fundamental estimate (5) does not hold for the norm $\|\cdot\|_N$.

In fact, it seems difficult, if not impossible, to establish an estimate of type (5) in so-called Banach limit spaces (Chapter III), such as $L^3(\mathbb{R}^3)$, $H^{1/2}(\mathbb{R}^3)$ or $M_2^3(\mathbb{R}^3)$. We show, following *T. Kato* [Math. Z. 187, 471–480 (1984; Zbl 0545.35073)], that it is not necessary to solve problem (5) in order to obtain an existence theorem for solutions of the Navier-Stokes equations in $\mathcal{C}([0, t); E)$, where E is a limit space.

Kato's idea is to add to the natural norm $\|\cdot\|_N$ the term $\sup_{0\leq t\leq T} t^{\alpha/2}\|\mathbf{v}(t, \cdot)\|_F$, where $\alpha > 0$ is an exponent and the second Banach space F seems unrelated to the Banach space E. The role played by F is to allow (5) to be satisfied, whereas this would not be the case (a priori) with the original definition. 'Kato's artificial norm', defined by (10) $\|\mathbf{v}\|_A = \|\mathbf{v}\|_N + \sup_{0\leq t\leq T} t^{\alpha/2}\|\mathbf{v}(t, \cdot)\|_F$, has the properties (11) $\|\mathbf{B}(\mathbf{v}, \mathbf{u})\|_A \leq \eta\|\mathbf{v}\|_A\|\mathbf{u}\|_A$ and (12) $\|S(t)\mathbf{v}_0\|_A \leq c\|\mathbf{v}_0\|_E$ for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{v}_0$ in E, with η and c two absolute constants. The introduction of the artificial norm also allows us to apply a fixed-point algorithm in what had seemed to be a hopeless situation.

More precisely, we obtain, thanks to the a priori estimates (11) and (12), an existence theorem for global solutions $\mathbf{v}(t, \mathbf{x}) \in \mathcal{C}([0, \infty); E)$, where E is a limit space and $\mathbf{v}_0$ is in E, which are divergence-free and satisfy (13) $\|\mathbf{v}_0\|_E \leq 1/4\eta c$, where η and c are defined by (11) and (12) (Theorem 3.2.1). The price we pay for using 'Kato's artificial norm' is that we no longer (necessarily) have uniqueness inside the natural space $\mathcal{C}([0, \infty); E)$.

After giving the proof of Kato's theorem, we show (Theorem 3.3.7) that this result can be greatly improved. Now it is the norm of $\mathbf{v}_0$, in a weaker suitable Besov space, that must be sufficiently small, which is the case if $\mathbf{v}_0$ is sufficiently oscillating. In other words, thanks to the viscosity, which acts especially on the high frequencies, rapid oscillations in the initial data can have a smoothing effect and provide a (regular) global solution $\mathbf{v}(t, \mathbf{x}) \in \mathcal{C}([0, \infty); E)$.

4. Finally, one will note that the limit spaces E (e.g., $L^3(\mathbb{R}^3)$, $H^{1/2}(\mathbb{R}^3)$, $M_2^3(\mathbb{R}^3)$) used in the study of the existence of global solutions $\mathbf{v}(t, \mathbf{x}) \in \mathcal{C}([0, \infty); E)$ are such that their norm is invariant under the normalized dilatations $\mathbf{v}(\mathbf{x}) \to \lambda\mathbf{v}(\lambda\mathbf{x})$, $\lambda > 0$. This property allows us to foresee the existence of self-similar solutions for the Navier-Stokes equations, i.e., solutions $\mathbf{v}(t, \mathbf{x})$ such that $\mathbf{v}(t, \mathbf{x}) = \lambda\mathbf{v}(\lambda^2 t, \lambda\mathbf{x})$, for all $\lambda > 0$.

In Chapter IV, which is devoted entirely to this subject, we give an affirmative answer to the problem (i.e., a global existence and uniqueness theorem) provided that one works in certain functional spaces (notably the Besov spaces) which contain homogeneous functions of degree -1 and whose norm is invariant under normalized dilatations.

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MSC:

35Q30	Navier-Stokes equations
35-02	Research exposition (monographs, survey articles) pertaining to partial differential equations
42C40	Nontrigonometric harmonic analysis involving wavelets and other special systems
76D03	Existence, uniqueness, and regularity theory for incompressible viscous fluids
76D05	Navier-Stokes equations for incompressible viscous fluids

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Littlewood-Paley decomposition; wavelet decomposition; Morrey-Campanato spaces; Besov spaces; Triebel-Lizorkin spaces; local existence and uniqueness; global existence and uniqueness; selfsimilar solutions