Cho, Jung R.; Ito, Noboru; Kim, Pan Soo

On Hadamard property of a certain class of finite groups. (English) Zbl 0911.20008

J. Algebra 204, No. 2, 666-674 (1998).

A finite group G of order $8n$ with a central involution e^* is called a Hadamard group if G contains a transversal D with respect to $\langle e^*\rangle$ such that $|D\cap Dr|=2n$ for every element r of G outside $\langle e^*\rangle$. Such a transversal is called a Hadamard subset. The cyclic group of order 4 is considered as a Hadamard group. If $r^2=e^*$, then we have $|D\cap Dr|=2n$ for every transversal D of G with respect to $\langle e^*\rangle$. Let $A(e^*)$ be the set of elements r of G such that $r^2=e^*$ and $a(e^*)=|A(e^*)|$. Then if $a(e^*)$ is large enough compared with $|G|$ we may expect G to be Hadamard. In the present paper one assumes that $a(e^*)\geq 4n$ and we investigate the Hadamard property of G (of order $8n$). Let $T(G)$ be the sum of the irreducible characters of G. Then $T(G)\leq a(e^*)$, by the Frobenius-Schur formula on the number of involutions. This means that $2T(G)\geq |G|$, and such groups were classified by the reviewer and K. G. Nekrasov [see *Ya. G. Berkovich* and *E. M. Zhmud*, Characters of finite groups, Part 1. Transl. Math. Monogr. 172, Am. Math. Soc., Providence (1998), Chapter 11]. The authors divide their groups into three classes: Class I consists of groups such that $a(e^*)>4n$, class II consists of groups G such that $a(e^*)=4n$ and G contains a nonreal element, and class III consists of groups G such that $a(e^*)=4n$ and all elements of G are real. For groups of classes I and II we have that $2T(G)>|G|$. Note that generalized quaternion groups are members of class I (it is known that these groups are Hadamard; semidihedral and dihedral groups are not Hadamard).

The main result of this paper is the following Proposition 3. Any 2-group of order $8n$ with $a(e^*)\geq 4n$ is Hadamard.

Note that there exists a non-Hadamard group among groups of class II. Namely, class II contains the series of groups $X(n)$ presented by $X(n)=\langle r,s\mid r^{2n}=s^4=1,\ s^{-1}rs=r^{-1}\rangle$.

Proposition 4. If $X(n)$ is Hadamard, then n is a sum of two squares. In particular, $X(3)$ is not Hadamard.

The class of Hadamard groups is very large. Indeed, as Ito showed, for every 2-group P there exists a Hadamard 2-group G such that P is isomorphic to a subgroup of G. All known groups of class I are Hadamard. Class I contains the series of groups $G(n)=\langle r,s\mid r^{2n}=s^2=e^*,\ s^{-1}rs=r^{-1}\rangle$. These groups were investigted in the following papers: *A. Baliga* and *K. J. Horadam* [Australas. J. Comb. 11, 123-134 (1995; Zbl 0838.05017)] and *D. L. Flannery* [J. Algebra 192, No. 2, 749-779 (1997; Zbl 0889.05032)]. There are many open questions on Hadamard groups (for example, abelian Hadamard groups are not classified).

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MSC:

20C15 Ordinary representations and characters
05B20 Combinatorial aspects of matrices (incidence, Hadamard, etc.)
20D60 Arithmetic and combinatorial problems involving abstract finite groups
05B10 Combinatorial aspects of difference sets (number-theoretic, group-theoretic, etc.)

Keywords:

finite groups; central involutions; transversals; irreducible characters; generalized quaternion groups; Hadamard groups; Hadamard 2-groups

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