Braverman, A.; **Etingof, P.**; **Gaitsgory, D.**

Quantum integrable systems and differential Galois theory. (English) Zbl 0901.58021

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Summary: This paper is devoted to a systematic study of quantum completely integrable systems (i.e., complete systems of commuting differential operators) from the point of view of algebraic geometry. We investigate the eigenvalue problem for such systems and the corresponding D-module when the eigenvalues are in generic position. In particular, we show that the differential Galois group of this eigenvalue problem is reductive at generic eigenvalues. This implies that a system is algebraically integrable (i.e., its eigenvalue problem is explicitly solvable in quadratures) if and only if the differential Galois group is commutative for generic eigenvalues. We apply this criterion of algebraic integrability to two examples: finite-zone potentials and the elliptic Calogero-Moser system. In the second example, we obtain a proof of the Chalyh-Veselov conjecture that the Calogero-Moser system with integer parameter is algebraically integrable, using the results of Felder and Varchenko.

MSC:

37J35 Completely integrable finite-dimensional Hamiltonian systems, integration methods, integrability tests
37K10 Completely integrable infinite-dimensional Hamiltonian and Lagrangian systems, integration methods, integrability tests, integrable hierarchies (KdV, KP, Toda, etc.)
81Q30 Feynman integrals and graphs; applications of algebraic topology and algebraic geometry
37N99 Applications of dynamical systems
14H99 Curves in algebraic geometry
12H05 Differential algebra
13N10 Commutative rings of differential operators and their modules

Keywords:

D-module; quantum completely integrable systems; eigenvalue problem; differential Galois group; algebraic integrability

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References:

AK A. Altman, S. Kleiman,Introduction to Grothendieck Duality Theory, Lecture Notes in Math., 146, Springer-Verlag, 1970. · Zbl 0215.37201

Am S. Amitsur,Commutative linear differential operators, Pacific Journal of Math.8 (1958), 1-11. · Zbl 0218.12054

Ar ?. ?. ??????,????????????? ?????? ???????????? ???????, ??????, ??????, ?????, 1974. English translation: V. Arnold,Mathematical Methods of Classical Mechanics, Graduate Texts in Math., 060, Springer-Verlag, 1978.

BD A. Beilinson and V. Drinfeld,Quantization of Hitchin's Hamiltonians and Hecke eigensheaves, Preprint (1996). · Zbl 0864.14007

Bo A. Borel et al.,Algebraic D-modules, Progress in Mathematics, Birkhäuser, Boston 1988.

CV1 O. Chalykh, A. Veselov,Commutative rings of partial differential operators and Lie algebras, Comm. Math. Phys.126 (1990), 597-611. · Zbl 0746.47025 · doi:10.1007/BF02125702

CV2 O. Chalykh, A. Veselov,Integrability in the theory of Schrodinger operator and harmonic analysis, Comm. Math. Phys.152 (1993), 29-40. · Zbl 0767.35066 · doi:10.1007/BF02097056

De P. Deligne,Catégories Tannakiennes, Grothendieck Festschrift2 (1991), Birkhäuser, Boston, 111-195.

DMN B. Dubrovin, V. Matveev, S. Novikov,Nonlinear equations of KdV type, finite zone linear operators, and abelian varieties, Uspekhi Mat. Nauk31 (1976), no. 1, 55-136 (in Russian). English translation: B. Dubrovin, V. Matveev, S. Novikov,Nonlinear equations of KdV type, finite zone linear operators, and abelian varieties, Russian Math. Surv.31 (1976), no. 1, 59-146.

Dr V. Drinfeld,Commutative subrings of certain noncommutative rings, Funktsional. Anal. i Ego and Prilozh.11 (1977), no. 1, 11-14 (in Russian). English translation: V. Drinfeld,Commutative subrings of certain noncommutative rings, Funct. Anal.

Appl.11 (1977), no. 1, 9-11.

EFK P. Etingov, I. Frenkel and A. Kirillov, Jr.,Spherical functions on affine Lie groups, Duke Math. J.80 (1995), no. 1, 59-90. · Zbl 0848.43010 · doi:10.1215/S0012-7094-95-08003-X

FV G. Felder and A. Varchenko,Three formulas for eigenfunctions of integrable Schrödinger operators, Preprint (1995).

Ka I. Kaplansky,An Introduction to Differential Algebra, Hermann, Paris, 1957. · Zbl 0083.03301

Katz N. Katz,On the calculation of certain differential Galois groups, Inv. Math.87 (1987), 13-62. · Zbl 0609.12025 · doi:10.1007/BF01389152

Kr I. Krichever,Methods of algebraic geometry in the theory of non-linear equations, Uspekhi Mat. Nauk32 (1977), no. 6, 182-214 (in Russian). English translation: I. Krichever,Methods of algebraic geometry in the theory of nonlinear equations, Russian Math. Surv.32 (1977), no. 6, 183-208.

ML L. Makar-Limanov,The automorphisms of the free algebra with two generators, Funktsional. Anal. i Ego Prilozh.4 (1970), no. 3, 107-108 (in Russian). English translation: L. Makar-Limanov,The automorphisms of the free algebra with two generators, Funct. Analysis and Appl.4 (1970), no. 3, 332-333. · Zbl 0218.13006

OP M. Olshanetsky and A. Perelomov,Quantum integrable systems related to Lie algebras, Phys. Rep.94 (1983), 313-404. · doi:10.1016/0370-1573(83)90018-2

VSC A. Veselov, K. Styrkas, O. Chalykh,Algebraic integrability for the Schrödinger equation and finite reflection groups, Teor. i Mat. Fiz.94 (1993), no. 2, 253-275 (in Russian). English translation: A. Veselov, K. Styrkas, O. Chalykh,Algebraic integrability for the Schrödinger equation and finite reflection groups, Theor. and Math. Phys.94 (1993), no. 2, 182-197.

WW E. T. Whittaker, G. N. Watson,Course of Modern Analysis, 4th Edition, Cambridge Univ. Press, 1958.