Shimura, Goro

**Abelian varieties with complex multiplication and modular functions.** (English)

Zbl 0908.11023
Princeton Mathematical Series. 46. Princeton, NJ: Princeton University Press. ix, 217 p. (1998).

The history of complex multiplication in the arithmetic theory of abelian varieties can be traced back to the fundamental works of Gauss and Abel on elliptic functions. However, as a subject of its own right, complex multiplication theory was initiated by Kronecker's approach to studying abelian extensions of imaginary quadratic number fields. In 1900, Hilbert proposed the generalization of Kronecker's approach as the twelfth of his famous problems and emphasized its importance for the progress in number theory. Basically, Hilbert's 12th problem stated that abelian extensions over any given algebraic number field should be constructible by means of special values of suitable analytic functions via their so-called "complex multiplication".

In the first half of this century, the main achievements in this direction were obtained by Hecke and, with regard to the special theory of complex multiplication of elliptic functions, by the works of Hasse, Deuring, and Weil. In the 1950s, *G. Shimura* and *Y. Taniyama* presented a new, very general approach to complex multiplication theory, which was based upon the rigorous framework of modern algebraic geometry, especially on A. Weil's abstract theory of abelian varieties. A full exposition of these new ideas, methods and results appeared in the monograph [|Complex multiplication of abelian varieties and its applications to number theory (1961; Zbl 0112.03502)].

The book under review is Professor Shimura's attempt at providing an up-dated version of his (and the late Y. Taniyama's) earlier, fundamental monograph from more than 35 years ago. In view of the tremendous progress made in the field since then, this is certainly a highly welcome and gratifying gift of G. Shimura's to the mathematical community.

In order to make this new book self-contained and homogeneous, the author has included the essential contents of the first sixteen sections of the 1961 monograph, followed by seventeen new sections presenting the more recent material. However, as to the old part covered by the first sixteen sections, the author has made considerable and thorough revisions, mainly so by stating some theorems in stronger forms and inserting explanatory, up-dating remarks throughout. The principal feature of the book is, of course, the new part, which mainly presents the progress made by the author himself, since 1961, in three important directions: the zeta function of an abelian variety with complex multiplication, families of abelian varieties and modular functions, and theta functions and periods of integrals on abelian varieties. The results in these topics discussed here were so far published in about 25 of the author's research articles between 1961 and 1994, and many of them are treated here, for the first time, in an extensive, comprehensive and systematic manner as an organic part of a homogeneous monograph.

As to the contents of the book, Chapters I–IV form the revisited old part from 1961. Chapter I provides the basic general material on abelian varieties, including differential forms, the analytic aspects of the theory and polarizations, whereas Chapter II is devoted to the algebraic part of the theory of complex multiplication. Chapter III provides the theory of reduction of algebraic varieties modulo prime divisors of the base field, and Chapter IV deals with the construction of class fields from fields of moduli of abelian varieties with given complex multiplication type. This chapter concludes with two new sections which are essential for the following generalizations of the classical theory of complex multiplication to higher-dimensional abelian varieties. These new sections discuss the field of moduli of a polarized abelian variety in a generalized setting and lead to a reformulation of the main theorem of complex multiplication in the adelic language.

The new Chapter V discusses the zeta function of an abelian variety with complex multiplication and its relations to certain Hecke $L$-functions. Chapter VI deals with families of abelian varieties parameterized by points on certain, hermitian symmetric spaces and with their relations to modular forms and functions on those spaces, whereas the concluding Chapter VII provides the author's comparatively new approach to investigating algebraic relations among the periods of abelian integrals on abelian varieties with complex multiplication. In the case of complex multiplication, these periods lead to new invariants, whose collection is called “the period symbol”, and the elaboration of the significance of those period symbols for the moduli theory of certain abelian varieties with complex multiplication is one of the highlights of G. Shimura’s new monograph under review.

Altogether, this is an important book on classical and new results in the fascinating theory of complex multiplication, written by one of the great contributors and pioneers himself, who was awarded the famous Leroy P. Steele Prize in 1996 for his lifetime achievement in mathematics, especially in the arithmetic theory of abelian varieties and automorphic functions by the American Mathematical Society.

Reviewer: W.Kleinert (Berlin)

**MSC:**

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| 11Fxx | Discontinuous groups and automorphic forms |
| 11-02 | Research exposition (monographs, survey articles) pertaining to number theory |
| 14-02 | Research exposition (monographs, survey articles) pertaining to algebraic geometry |
| 14K10 | Algebraic moduli of abelian varieties, classification |
| 14K20 | Analytic theory of abelian varieties; abelian integrals and differentials |
| 14K22 | Complex multiplication and abelian varieties |
| 14K25 | Theta functions and abelian varieties |
| 14G10 | Zeta functions and related questions in algebraic geometry (e.g., Birch-Swinnerton-Dyer conjecture) |
| 14G15 | Finite ground fields in algebraic geometry |



**Keywords:**

elliptic curves; complex multiplication; abelian varieties; zeta function; modular functions; theta functions; periods of integrals; class fields; field of moduli of a polarized abelian variety; Hecke *L*-functions; periods of abelian integrals; period symbol; differential forms; polarizations