Milgram, R. J.

The structure of spaces of Toeplitz matrices. (English) Zbl 0894.57016

Topology 36, No. 5, 1155-1192 (1997).

A finite Toeplitz matrix is a matrix $t_n = (a_{i,j})$ with the coefficients of the form $a_{i,j} = a_{i-j}$. Consider the variety of projective equivalence classes of nonzero $n \times n$-Toeplitz matrices ($t_n \equiv \alpha t_n$ for all nonzero $\alpha \in \mathbb{C}$), which are parameterized by the complex projective space $\mathbb{C}P^{2n-2}$. The paper is devoted to a study of subvarieties $T_{n,k}$ consisting of those t_n with $\dim(\mathrm{kernel}(t_n)) \geq k$, and the open varieties $\mathcal{T}_{n,k} = T_{n,k} - T_{n,k+1}$. The results are summarized in the following three theorems.

Theorem A. The space $\mathcal{T}_{n,0}$ of nonsingular $n \times n$-Toeplitz matrices is homeomorphic to the orbit space under the action of $GL_2(\mathbb{C})$ on the space of pairs $(p_1(z), p_2(z))$ of coprime polynomials with $\max(\deg(p_1(z)), \deg(p_2(z))) = n$, where the action is given as

$$(p_1\ p_2)\begin{pmatrix} a & c \\ b & d \end{pmatrix} = (ap_1 + bp_2\ cp_1 + dp_2).$$

Theorem B. Let $t_n \in \mathcal{T}_{n,k}$. Then there is a vector $w \in \mathrm{Ker}(t_n)$ so that

$$\mathrm{Ker}(t_n) = \langle w, s(w), \dots, s^{k-1}(w) \rangle,$$

where s is a shift operator. Moreover, (1) the first $k-1$ coordinates of w are zero; (2) w is unique up to multiplication by a nonzero scalar.

Theorem C.

$$H^*(\mathcal{T}_{n,k}; \mathbb{Q}) \cong H^*(S^2; \mathbb{Q}), k \geq 1$$
$$H^*(\mathcal{T}_{n,0}; \mathbb{Q}) \cong H^*(pt; \mathbb{Q}).$$

The rational cohomology of the unprojectivized versions of $\mathcal{T}_{n,k}$ are also computed. These results lead to a complete determination of the rational cohomology of the strata in a stratification of the moduli spaces $\mathcal{M}_k$ of gauge equivalence classes of SU(2)-Yang-Mills instantons on S^4. This stratification was described in [*C. P. Boyer, J. C. Hurtubise, B. M. Mann* and *R. J. Milgram*, Ann. Math., II. Ser. 137, No. 3, 561-609 (1993; Zbl 0816.55002)].

Reviewer: I.Itenberg (Rennes)

MSC:

57N65 Algebraic topology of manifolds
32G13 Complex-analytic moduli problems
55P35 Loop spaces
58D27 Moduli problems for differential geometric structures

Keywords:

space of Toeplitz matrices; moduli space; instantons; stratification; Toeplitz matrix

Full Text: DOI