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Nearly abelian, nilpotent, and Engel lattice-ordered groups. (English) Zbl 0856.06011

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From the authors' abstract: "In this paper, we examine those classes of lattice-ordered groups in which every substitution produces a group element comparable to the group identity, and, under certain natural conditions, obtain a description of the structure of such lattice-ordered groups in terms of the radical of the corresponding ℓ-variety. We especially concentrate on those sets of words which produce the ℓ-varieties of Abelian, nilpotent, Engel, and solvable lattice-ordered groups."

In more detail: An ℓ-group G is nearly-ν with respect to an equational basis $\{w_n(\vec{x})\}$ of an ℓ-variety ν of ℓ-groups if for any substitution $\vec{x} \to \vec{g}$ into G, $w_n(\vec{g}) \diamondsuit e$ ($a \diamondsuit b$ denotes that a is comparable to b). An ℓ-group word $w(\vec{x})$ is balanced if any ℓ-group G satisfies the condition that for any substitution $x_{ijk} \to g_{ijk}$ into G with $w(\vec{g}) > e$ there exists a substitution $x_{ijk} \to h_{ijk}$ into the ℓ-subgroup generated by $\{g_{ijk}\}$ such that $w(\vec{h}) < e$, and vice versa. An ℓ-group word that is not balanced will be called unbalanced. An ℓ-variety ν has a balanced basis if there exists an equational basis of balanced ℓ-group words for ν. It is not known whether every ℓ-variety has a balanced basis. The following theorem holds: Let ν be a normal-valued ℓ-variety of lattice-ordered groups with equational basis $\{w_\lambda(\vec{x})\}$. Let G be nearly-ν with respect to $\{w_\lambda(\vec{x})\}$ and Δ be a normal plenary subset of $\Gamma(G)$. Then G can be ℓ-embedded into a special-valued ℓ-group H that is also nearly-ν with respect to $\{w_\lambda(\vec{x})\}$.

Let us denote by A, N_k, E_k and A_k, the following ℓ-varieties: Abelian, nilpotent of class k, Engel of bound k, solvable of rank k, respectively, and by L_k the powers of the Abelian ℓ-varieties A^k. For these ℓ-varieties let us refer to the following as their canonical bases: $A : [x, y] = e$; $N_k : [x_1, \dots, x_{k+1}] = e$; $E_k : [x, y, \dots, y]_k = e$ (the repeated commutator with k occurrences of y); $L_k : [x^k, y^k] = e$. The equational bases for A^k are built up recursively by letting $w_1(y, z) = [x_1, x_2]$ and

$$w_{k+1}(x_{2k-1}, x_{2k}, w_k(\vec{y}), w_k(\vec{z})) = [|x_{2k-1}| \wedge |w_k(\vec{y})|, |x_{2k}| \wedge |w_k(\vec{z})|];$$

then the canonical basis for A^k is $w_k(\vec{x}) = e$. An obviously similar recursion exists for an equational basis for ℓ-groups that are solvable of rank k.

Theorem. The canonical bases of the ℓ-varieties N_k, L_k, A^k, A_k (for a positive integer k) and for the Engel ℓ-varieties E_2 and E_3 are balanced. It is not known whether the canonical basis for E_k, $k > 3$, is balanced. We only know that an ℓ-group G is representable if $[a, b, \dots, b]_n \diamondsuit e$ for all $a, b \in G$.

There are some more details and an example in which it is shown: If ℓ-group words $w_1(\vec{x})$ and $w_2(\vec{x})$ generate the same ℓ-variety ν, then an ℓ-group G that is nearly-ν with respect to $w_1(\vec{x})$ need not be nearly-ν with respect to $w_2(\vec{x})$, even when both ℓ-group words $w_1(\vec{x})$ and $w_2(\vec{x})$ are balanced.

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MSC:

06F15 Ordered groups
20F18 Nilpotent groups
20F45 Engel conditions
20F60 Ordered groups (group-theoretic aspects)

Keywords:

special-valued ℓ-group; normal-valued ℓ-group; nearly-ν ℓ-group; lattice-ordered groups; ℓ-variety; Abelian; nilpotent; Engel; solvable; ℓ-group word; balanced; equational basis; canonical bases