Cassels, J. W. S.; **Flynn, E. V.**

Prolegomena to a middlebrow arithmetic of curves of genus 2. (English) Zbl 0857.14018

London Mathematical Society Lecture Note Series. 230. Cambridge: Cambridge Univ. Press. xiv, 218 p. (1996).

The monograph may be regarded as a natural generalization of the thirty years old article "Diophantine equations with special references to elliptic curves", J. Lond. Math. Soc. 41, 193–291 (1966; Zbl 0138.27002)] by *J. W. S. Cassels*, to genus 2 curves and their Jacobians. Arithmetic (i.e., number theoretic properties) of genus 2 curves over fields (e.g., $\mathbb{Q}$, number fields, or local fields) are explored, generalizing the methods and techniques, such as group laws, 2-descents, formal groups, torsion points, heights, a weak Mordell-Weil theorem, etc., developed for elliptic curves to genus 2 curves and their Jacobians (or rather the associated Kummer surfaces).

It might be said that the principal goal of the monograph is to compute the Mordell-Weil groups of genus 2 curves defined over $\mathbb{Q}$. Indeed, the Mordell-Weil groups (at least the rank) are determined explicitly for a large number of genus 2 curves.

Let k be a perfect field of characteristic $\neq 2$. A curve of genus 2 defined over k is shown to be birationally equivalent to a curve in a canonical form $\mathcal{C}: Y^2 = F(X)$ where $F(X) = f_0 + f_1X + f_2X^2 + f_3X^3 + f_4X^4 + f_5X^5 + f_6X^6 \in k[X]$ has no multiple factors. The Jacobian variety $J(\mathcal{C})$ of $\mathcal{C}$ is constructed, as a surface in 15-dimensional projective space $\mathbb{P}^{15}$ by computing explicitly the 16 basis elements defining $J(\mathcal{C})$. The group law on $J(\mathcal{C})$ is described generically. The Jacobian $J(\mathcal{C})$ is unfortunately too large for any practical and computational purposes. To remedy this difficulty, the associated Kummer surfaces $\mathcal{K} = \mathcal{K}(\mathcal{C})$ and their Jacobians are investigated in detail. The Kummer surfaces are quartic surfaces in $\mathbb{P}^3$ determined by four of the 16 basis elements, and they contain much of the information on the Jacobians. Unfortunately, the group structures on the Jacobian $J(\mathcal{C})$ will be lost in passing from the Jacobians to the Kummer surfaces, though addition on 2-division points still remain to be significant on the Kummer surfaces.

The classical result that every abelian variety of dimension 2 (over an algebraically closed field) is isogenous to the Jacobian of a genus 2 curve is generalized to abstract Kummer surfaces defined over $\mathbb{Q}$.

The computation of the Mordell-Weil group $\mathfrak{G}$ of $J(\mathcal{C})$ is one of the principal results of the monograph, when the field k is a number field. Analogous to genus 1 case, the weak Mordell-Weil theorem asserting the finiteness of the group $\mathfrak{G}/2\mathfrak{G}$ is proved. This is done by constructing a group homomorphism with kernel $2\mathfrak{G}$ from $\mathfrak{G}$ to an easily treated group. For this the Kummer surfaces as well as the dual Kummer surfaces are used. Then the Mordell-Weil theorem that $\mathfrak{G}$ is finitely generated is established, invoking the use of a height function on the jacobian (as in the genus 1 case).

The torsion group of $J(\mathcal{C})$ is discussed imposing two concrete problems: given a genus 2 curve, (1) how can one find the rational torsion group in $J(\mathcal{C})$? and (2) given an integer N, how does one try to find a curve whose Jacobian has a rational point of order N? In an attempt to answer these questions, the group law, the formal group, and isogenies are explicitly described. Then a crude algorithm which might lead to a complete solution to the question (1) is presented. For (2), a method for finding large torsion elements over $\mathbb{Q}$ are obtained, and computation of torsion elements of orders ≤ 29 are carried out for a number of genus 2 curves. The actual computation of the Mordell-Weil group is carried out by performing complete 2-descents on several genus 2 curves. Most of the examples discussed here have Mordell-Weil rank 0 or 1, with small torsion subgroups. For instance, $\mathcal{C}: Y^2 = X(X-1)\,(X-2)\,(X-5)\,(X-6)$ has the Mordell-Weil rank 1 with $\mathfrak{G}/2\mathfrak{G} = \langle \mathfrak{G}_{\text{tors}}, \{(3,6), \infty\}\rangle$ and $\mathfrak{G}_{\text{tors}}$ consisting only of the 2-torsion group of order 16. Curves with large Mordell-Weil rank are briefly discussed.

Other results are concerned with finding all the rational points on genus 2 curves, the number of which is known to be finite by a theorem of Faltings. Here the authors make use of Chabauty's theorem to find rational points on a genus 2 curve. One of worked out examples is the curve $\mathcal{C}: Y^2 = (X^2 - 2X - 2)\,(-X^2 + 1)\,(2X)$, all whose rational points are given by $\mathcal{C}(\mathbb{Q}) = \{(0,0), \infty, (\pm 1, 0), (-1/2, \pm 3/4)\}$. – The endomorphism ring of the Jacobian which is strictly larger than $\mathbb{Z}$ is described for some genus 2 curves. These give rise to examples of genus 2 curves with complex, or real multiplication.

The monograph consists of 18 chapters, some theoretical, and others computational. For instance, the chapters 8, 10, 11, 12, 13 are of computational nature. The contents are: Chapter 1: Curves of genus 2; Chapter 2: Constructions of the Jacobian; Chapter 3: The Kummer surface; Chapter 4: The dual of the Kummer surface; Chapter 5: Weddel's surface; Chapter 6: $\mathfrak{G}/2\mathfrak{G}$; Chapter 7: The Jacobian over local fields – formal groups; Chapter 8: Torsion; Chapter 9: The isogeny – theory; Chapter 10: The isogeny – applications; Chapter 11: Computing the Mordell-Weil group, Chapter 12: Heights, Chapter 13: Rational points – Chabauty's theorem: Chapter 14: Reducible Jacobians; Chapter 15: The endomorphism rings; Chapter 16: The desingularized Kummer surface; Chapter 17: A neoclassical approach; and Chapter 18: Zukunftsmusik. In the appendix, MAPLE programs and other files available by anonymous ftp are listed.

There are numerous worked out examples of genus 2 curves, demonstrating the power of computer algebra as a research tool in diophantine equations, especially, for genus 2 curves.

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MSC:

14H45 Special algebraic curves and curves of low genus
11G30 Curves of arbitrary genus or genus $\neq 1$ over global fields
14H40 Jacobians, Prym varieties
11-02 Research exposition (monographs, survey articles) pertaining to number theory
14-02 Research exposition (monographs, survey articles) pertaining to algebraic geometry
14H25 Arithmetic ground fields for curves
14G05 Rational points
14G40 Arithmetic varieties and schemes; Arakelov theory; heights
14Q05 Computational aspects of algebraic curves
11Y50 Computer solution of Diophantine equations

Keywords:

Mordell-Weil groups; genus 2 curves; Jacobian variety; Kummer surfaces; torsion group; rational points; computer algebra; Diophantine equations

Software:

Maple