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Grundprobleme der mathematischen Theorie elektromagnetischer Schwingungen. (German)

Zbl 0087.21305

Die Grundlehren der mathematischen Wissenschaften. Bd. 88. Berlin-Göttingen-Heidelberg: Springer Verlag. 360 S., 8 Abb. (1957).

Electromagnetic theory has in recent years tended to become a no man's land, being viewed by the physicist as mathematics and by the mathematician as physics. The main emphasis, alike in textbooks and in research, has been on special problems, particularly those involving bodies of special forms. Of course neither the practical importance, nor the continuing mathematical stimulation of such problems is to be overlooked. Nevertheless, electromagnetic theory poses other problems, and it should be emphasized at the outset that the book under review does not deal at all with problems which belong now in the field of technology.

Comparatively few have perceived that the emphasis in electromagnetic theory on special problems, soluble by such devices as the separation of variables, leaves a substantial gap in the mathematical side of the theory. One has in mind here such topics as the behaviour of fields in arbitrarily inhomogeneous media, their asymptotic behaviour at infinity. and the wide fields of boundary problems concerning bodies of arbitrary shape. There is here a striking contrast with the development of potential theory. for which a rigorous and thorough study has long been an accepted branch of pure mathematics. Without minimizing the early contributions of Sommerfeld and Weyl, it has been mainly the author's achievement to have filled this gap. So far as existence theorems for boundary problems are concerned, there are of course important parallels in independent work of the Georgian school, in particular of Kupradze and Avazašvili (see *V. D. Kupradze*'s book [Randwertaufgaben der Schwingungstheorie and Integralgleichungen. Berlin: VEB Deutscher Verlag der Wissenschaften (1956; Zbl 0070.09701)]) who considered also problems of elasticity.

The present work is devoted to a unified presentation of the author's work. In addition to results of a substantial character. such as those dealing with boundary problems. he has provided a much-needed rigorous account of the foundations. It is clear, for example, that vector analysis involves such matters as continuity and differentiability and deserves to be treated in the spirit of the theory of functions of a real variable. Again, the discussion of fields and currents associated with a surface of fairly general shape involves topics from differential geometry and topology. In introducing a new standard of rigour, the book is analogous to *O. D. Kellogg*'s "Foundations of Potential Theory" [Berlin: Springer (1929; JFM 55.0282.01)].

However, electromagnetic theory provides a richer field of problems, which are usually more difficult than their potential-theoretic analogues, when indeed they have one. The work opens with a review of Maxwell's equations; here and throughout the time does not appear explicitly, being accounted for by a factor $e^{-i\omega t}$. Lest one be tempted to think that electromagnetic theory is no more than a branch of the topic of partial differential equations, a difference appears in that the integral forms of Maxwell's equations turn out to be more fundamental than the differential forms. The upshot of this is that greater generality is attained if the vector differential operators are defined not by an explicit coordinate representation, but rather directly by the limiting process suggested by the theorems of Gauss and Stokes. When the explicit coordinate forms of these operators are valid, for example, when the functions are continuously differentiable, they of course coincide with the generalized operators; however the converse is not unrestrictedly true. Such questions are investigated in the first chapter, together with the extension to the generalized operators of standard theorems; as was shown in an earlier author's paper [Math. Ann. 124, 427–449 (1952; Zbl 0047.40101), the restrictions usually imposed can be substantially weakened. Also considered in this chapter are "regular" surfaces, used for the integral theorems. These are in general continuously differentiable, but may have suitably restricted edges and corners.

The second chapter deals with Legendre and Bessel functions, derived by way of the Helmholtz wave equation. Although dealing here with a well-worn theme, the author has been able to present these topics in a fresh and interesting way. As befits the application to the wave equation, the Bessel functions used are essentially Hankel functions, though the author uses his own notation and derives the properties ab initio.

Chapter III, on the Helmholtz equation, deals with two main topics. There is firstly the problem of the asymptotic behaviour, at great distances, of solutions of $\Delta u + k^2 u = 0$, where k is constant. The author gives the theory of the Sommerfeld radiation condition, and also his theory of radiation patterns, according to which the behaviour at great distances is asymptotically given in terms of a harmonic function, which determines the field. The other section of this chapter relates to $\Delta u + k^2 u = 0$, where k is variable. One result is that there is a function k such that any twice differentiable solution u must vanish identically. Another interesting result gives $U = \partial U/\partial n = 0$ as boundary conditions which ensure that a solution of $\Delta u + k^2 u = 0$ in a region vanishes; this is of course nearly trivial if k is constant.

The very substantial Chapter IV, gives a thorough analysis of solutions of Maxwell's equations in homogeneous media, bounded on the inside or outside by a regular surface. The main objects of study are representations of $\mathfrak{E}$ and $\mathfrak{E}$ as surface or volume integrals. It is necessary to study surface integrals either in terms of boundary values, or with arbitrarily prescribed integrands, satisfying for example Hölder or continuity restrictions, or having specified singularities. Also in this chapter is the requisite geometrical study of surface, for which tensors are introduced.

As a preparation for the use of integral equations in connection with boundary problems the author gives in Chapter V, a full account of the Riesz theory of the completely continuous operator. To begin with, this follows closely the original paper of *F. Riesz* [Acta Math. 41, 71–98 (1916; JFM 46.0635.01)]. Later on, however, adjoint operators, an inner product, and the concurrent use of two norms are introduced. The applicability of the general concepts to specific integral operators over surfaces is then dealt with.

Some of the main results of the theory are to be found in Chapter VI, which gives existence theorems for three diffraction problems. In all of them there is to be a field satisfying the radiation condition at infinity, and satisfying Maxwell's equations in space which is homogeneous except for two finite regions. One such region may be considered as the source, and in it there are prescribed currents, both electric and magnetic. The second region occasions the diffraction, and the three problems differ as to whether ε and μ are constant in it, or vary continuously, the remaining possibility being that of perfect conductivity. Also interesting is a subsidiary result to the effect that a solution of

$$\nabla \times \mathfrak{H} + i\omega\varepsilon\mathfrak{E} = 0, \quad \nabla \times \mathfrak{E} - i\omega\varepsilon\mathfrak{H} = 0$$

in a region G bounded by a regular surface F, vanishes identically subject to the boundary conditions

$$\mathfrak{n} \times \mathfrak{E} = \mathfrak{n} \times \mathfrak{H} = 0 \text{ on } F;$$

here ε and μ are to be continuously differentiable. One may perhaps hope for a shorter proof of this result, and likewise of its scalar analogue, than that given here.

Chapter VII gives further basic existence theorems, again due to the author, but of a simpler character. It is a question of the existence of a field for which $\mathfrak{n} \times \mathfrak{E}$ takes assigned boundary values. In the case of the exterior problem the radiation condition is of course to be imposed.

Finally, there is a brief chapter on radiation patterns for solutions of Maxwell's equations, it being a question of the leading term $r^{-1}e^{i\omega r}\mathfrak{F}(\mathfrak{x}_0)$ in the expression for $\mathfrak{E}$ when r is large, where $\mathfrak{x}_0$ is a unit vector giving the direction, and $\mathfrak{F}(\mathfrak{x}_0)$ is an entire harmonic vector-field. In Chapter III the author has found a characterisation of such $\mathfrak{F}$). He now discusses the existence of directions in which the field may asymptotically polarised linearly or circularly, topological considerations being brought into play.

The bibliography lists some important texts, references to papers being confined to footnotes in the body of the book. In addition to the usual index, there is a special index for theorems, lemmas and definitions. The standard of book production is high, and misprints seem to be few and unimportant. The author is to be congratulated on his pioneer work. One may hope that it will stimulate generalizations, for example to non-steady-state phenomena in electromagnetism, and into the field of systems of partial differential equations.

Reviewer: F. V. Atkinson

For a scan of this review see the web version.

MSC:

78-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to optics and electromagnetic theory
78A25 Electromagnetic theory (general)
78A40 Waves and radiation in optics and electromagnetic theory

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