Cox, David A.

The homogeneous coordinate ring of a toric variety. (English) Zbl 0846.14032

J. Algebr. Geom. 4, No. 1, 17-50 (1995); erratum ibid. 23, No. 2, 393-398 (2014).

Complex, quasi-smooth, projective toric varieties may be considered a generalization of the projective space $\mathbb{P}^n$; they are obtained by glueing together affine pieces almost isomorphic to $\mathbb{C}^n$. Those toric varieties may be given by a fan (a certain collection of rational, polyhedral cones in $\mathbb{R}^n$) containing all combinatorial information necessary for this process. In the present paper, the author describes a different method of synthesizing projective varieties as geometric quotients from the given fan. The relations to projective spaces are even more striking: First, assigning to each one-dimensional generator of the fan Δ a coordinate, we obtain the affine space $\mathbb{C}^{\Delta(1)}$ (in case of $\mathbb{P}^n$, this will be $\mathbb{C}^{n+1}$). Then, we have to do the following two jobs simultaneously:

(i) Construct a certain subgroup of $(\mathbb{C}^*)^{\Delta(1)}$ by using the exact knowledge of the one-dimensional cones in Δ. This subgroup is isomorphic to some $(\mathbb{C}^*)^k$ and clearly acts on $\mathbb{C}^{\Delta(1)}$. (In case of $\mathbb{P}^n$, we have $k = 1$.)

(ii) Similarly to the construction of the Stanley-Reisner ring from a simplicial complex, the information which of the $\Delta(1)$-rays belongs to a common higher-dimensional cone (and which not) define a certain closed algebraic subset $Z \subseteq \mathbb{C}^{\Delta(1)}$ of codimension at least two. (Z equals $\{0\}$ in case of $\mathbb{P}^n$.)

Now, the main result is that the toric variety assigned to a simplicial fan Δ equals the geometric quotient $[\mathbb{C}^{\Delta(1)} \setminus Z]/(\mathbb{C}^*)^k$. Since the group action is defined without using the information about incidences of Δ-cones, this description is very useful for studying flips and flops, i.e. for changing Δ without changing $\Delta(1)$.

Finally, this result is used for studying the automorphism group of a toric variety via considering $\mathbb{C}^{\Delta(1)} \setminus Z$. In a paper of *Daniel Bühler* (Diplomarbeit Zürich), this is generalized also to non-simplicial fans.

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MSC:

14M25	Toric varieties, Newton polyhedra, Okounkov bodies
14L30	Group actions on varieties or schemes (quotients)
55U10	Simplicial sets and complexes in algebraic topology
14E07	Birational automorphisms, Cremona group and generalizations
14M17	Homogeneous spaces and generalizations

Keywords:

geometric quotients of a fan; projective toric varieties; Stanley-Reisner ring; automorphism group