Evans, Lawrence C.; Gariepy, Ronald F.

Measure theory and fine properties of functions. (English) Zbl 0804.28001

Studies in Advanced Mathematics. Boca Raton: CRC Press. viii, 268 p. (1992).

The purpose of this book is to present a theory of Radon measure and Hausdorff measure in $\mathbb{R}^n$ and some applications of the integrals with respect to them, in particular of some function classes for which certain formulas of calculus are valid under much more general conditions than usually.

Chapter 1 deals with the general theory of (outer) measure, measurable sets and functions, product measures with Fubini's theorem, Borel regular measures and Radon measures in $\mathbb{R}^n$, differentiation of a Radon measure with respect to another, and includes the Riesz representation theorem. Chapter 2 gives an introduction to the theory of Hausdorff measures in $\mathbb{R}^n$ together with the proof of the coincidence of the Hausdorff measure H^n with the Lebesgue measure L^n in $\mathbb{R}^n$. In Chapter 3, the formulas are studied that are connected with the Hausdorff measure of the image of a set under a Lipschitz mapping $f:\mathbb{R}^n\to\mathbb{R}^m$; in particular, we find, for $m\leq n$, the method of calculating $H^m(f(A))$ with the help of an L^n integral over A, as well as, for $m\geq n$, the curvilinear generalization of Fubini's theorem. Chapter 4 presents the theory of the Sobolev spaces $W^{1,p}(U)$ for an open set U in $\mathbb{R}^n$ and $1\leq p\leq\infty$. We find various approximation theorems, important inequalities (Gagliardo-Nirenberg-Sobolev, Poincaré, Morrey), and an introduction to the theory of capacity, together with its application to Sobolev functions. In Chapter 5, the authors discuss the theory of BV functions and of the sets with a BV characteristic function (= sets of finite perimeter); the latters are precisely those which admit a generalization of the Gauss-Green theorem. We find here generalizations to BV functions of some facts about Sobolev functions and the theorems that show the relation of BV functions to those of bounded variation in the classical sense, respectively to the sets with finite Hausdorff measure H^{n-1} in $\mathbb{R}^n$. Chapter 6 discusses theorems on approximate differentiability of BV functions, Alexandrov's theorem for the existence of a second derivative of convex functions, Whitney's extension theorem for functions with continuous first derivative, and its application for approximation theorems of Lipschitz, Sobolev, and BV functions.

A list of references (mostly textbooks and monographs), a short indication of the sources used in each chapter, a list of notations, and an index are added.

The authors make an effort to facilitate the reader's task by working out in great detail all calculations which are, in most books, left to the reader. On the other hand, they suppose that the reader is in possession of wide knowledge in analysis, functional analysis, and linear algebra; besides that, he has to find the motivation of a lot of steps in the proofs (e.g., to find out why certain sets or functions are measurable) and to check that a series of results can be used under weaker hypotheses than given in the book (e.g., the Riesz representation theorem is stated and proved for the space $C_0(\mathbb{R}^n)$ (= functions continuous in $\mathbb{R}^n$ with compact support), but later it is applied to the space $C_0(U)$ for open sets U in $\mathbb{R}^n$; in the definition of BV functions, a certain condition must be fulfilled for every function of $C_0^1(U)$ (= functions having continuous first derivative in an open set $U\subset\mathbb{R}^n$ and having compact support $K\subset U$), but later this is checked only for functions of $C_c^2(U)$ (= the above functions with continuous second derivative). However, a reader having ability, patience and time in order to fill up all these gaps, can become acquainted with a great amount of interesting methods and important results of modern real analysis.

Reviewer: Á.Császár (Budapest)

MSC:

28-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to measure and integration
28A75 Length, area, volume, other geometric measure theory
28A78 Hausdorff and packing measures
26B15 Integration of real functions of several variables: length, area, volume
26B20 Integral formulas of real functions of several variables (Stokes, Gauss, Green, etc.)
26B25 Convexity of real functions of several variables, generalizations

Keywords:

textbook; Radon measures; Hausdorff measures; capacity; Sobolev functions; BV functions; sets of finite perimeter; approximate differentiability; convex functions