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Geometrical finiteness for hyperbolic groups. (English) Zbl 0789.57007

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A complete orientable hyperbolic n-manifold arises as the quotient of hyperbolic n-space $\mathbf{H}^n$ by a torsion-free discrete group of orientation-preserving isometries of $\mathbf{H}^n$.

A fundamental notion in dimension 3 is that of geometrical finiteness. There are several equivalent definitions of geometrical finiteness. A classical definition is that $M = \mathbf{H}^3/G$ is geometrically finite if G admits a finite sided convex fundamental polyhedron in $\mathbf{H}^3$. A second definition of geometrical finiteness in dimension 3 due to Beardon and Maskit [*A. F. Beardon* and *B. Maskit*, Acta Math. 132, 1-12 (1974; Zbl 0277.30017)] concerns the nature of the limit points of elements of G, and a third definition due to Thurston is in terms of the thick part of the convex core of M – see below.

In higher dimensions one can consider extensions of these definitions, and this has been widely studied, see for example [*B. N. Apanasov*, Ann. Global Anal. Geom. 1, No. 3, 1-22 (1983; Zbl 0531.57012)], and [*P. Tukia*, Publ. Math., Inst. Hautes Étud. Sci. 61, 171-214 (1985; Zbl 0572.30036)], however accounts given are far from being complete. Indeed the standard generalization to $\mathbf{H}^n$ of the classical notion of geometrical finiteness in dimension 3 is not a natural one, so it is sensible to look for alternative definitions. As the author points out, at the time of writing it is an open question in general, as to whether a geometrically finite group in higher dimensions admits a finite sided convex fundamental domain. The author gives examples where no Dirichlet domain is finite sided.

The main purpose of the present paper is to provide a considerably detailed exposition of the equivalence of various natural generalizations of the notions of geometrical finiteness in dimension 3. The main theorem asserts that five definitions of geometrically finite are equivalent. Below we give rough statements of these.

Proofs are reasonably self contained and described carefully, since the history of this subject indicates a certain penchant for error.

We now briefly describe the notions of geometrical finiteness. In what follows M^n will always denote the quotient of hyperbolic n-space by a discrete torsion-free subgroup G of $\mathrm{Isom}^+(\mathbf{H}^n)$. In addition to avoid laboring with too much preliminary definitions we refer the reader to the paper for complete definitions of all terms.

Definition GF1: Let Ω the domain of discontinuity of G, and adjoin this to M to form $M_C(G) = (\mathbf{H}^n \cup \Omega)/G$. G is geometrically finite if $M_C(G)$ is the union of a compact set together with finitely many topological ends each of which corresponds to a maximal parabolic subgroup of G.

Definition GF2: G is geometrically finite if the limit set of G consists entirely of conical limit points and bounded parabolic fixed points.

Definition GF3: M^n is geometrically finite if it admits a representation as a finite polyhedral complex – this means essentially that G has a fundamental domain which is a finite union of polyhedra, each with a finite number of faces.

To state the final two definitions some notation is required. M is as above. $\mathrm{core}(M)$ denotes the convex core of M, that is the smallest submanifold with convex boundary whose inclusion into M is a homotopy equivalence. $\mathrm{thin}_\varepsilon(M)$ denotes the set of points of M where the injectivity radius is at most $\varepsilon/2$ and $\mathrm{thick}_\varepsilon(M)$ is defined to be the closure of $M \setminus \mathrm{thin}_\varepsilon(M)$. Finally, given a positive number η $N_\eta(\mathrm{core}(M))$ denotes an η neighbourhood of $\mathrm{core}(M)$. With this,

Definition GF4: G is geometrically finite if $\mathrm{core}(M) \cap \mathrm{thick}_\varepsilon(M)$ is compact.

Definition GF5: G is geometrically finite if for some η $N_\eta(\mathrm{core}(M))$ has finite volume.

These definitions apply to the case where G has elements of finite order, although a condition on finite subgroups is required in GF5 (although conjectured to be superfluous by the author). The author also gives a complete account in higher dimensions of geometrically finite groups which admit a finite sided Dirichlet domain. The main problem is to understand the cusps of geometrically finite manifolds in higher dimensions.

Reviewer: A.Reid (Cambridge)

MSC:

57M50	General geometric structures on low-dimensional manifolds	Cited in **2** Reviews
53C30	Differential geometry of homogeneous manifolds	Cited in **66** Documents

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hyperbolic n-manifold; geometrically finite; thick part of the convex core; domain of discontinuity; limit set; conical limit points; parabolic fixed points; fundamental domain; convex core; finite volume; Dirichlet domain; cusps

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