Gabai, David

Convergence groups are Fuchsian groups. (English) Zbl 0785.57004

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Between 1927 and 1942, Nielsen wrote a series of long papers of diffeomorphisms of surfaces. In recent years, Thurston completed Nielsen's theory of surface diffeomorphisms looking at it from the more general point of view of Teichmüller theory and modular groups. The last paper of the above series appeared in 1942 and concerned a problem which became later known as the Nielsen realization problem: given a finite group G of isotopy classes of diffeomorphisms of a surface $\mathcal{F}$ (a finite subgroup of the mapping class group π_0 Diff$(\mathcal{F}) \cong$ Out $\pi_1\mathcal{F}$), can one choose representatives realizing G as a group of diffeomorphisms of the surface? Nielsen's methods give a solution to this problem if there exists a "simple axis" for G: an isotopy class of a simple closed curve whose images under G give disjoint isotopy classes. The method is as follows: suppose $\mathcal{F}$ is covered by the hyperbolic plane $\mathbf{H}^2 = \text{Int } D^2$ (the interior of the 2-disk) and let F be the covering group which is a Fuchsian group (a discrete group of Möbius transformations). Purely algebraically, the group G defines an extension $1 \to F \hookrightarrow E \to G \to 1$, and by lifting representatives of elements of G to D^2 and restricting to $S^1 = \partial D^2$ one gets a well defined action of E on S^1 (but not on D^2 where only F acts). The problem then is to extend the action of G on S^1 to an action on D^2 extending the given action of F. If there exists a simple axis for G one may first extend to it preimages in $\mathbf{H}^2$ (a collection of disjoint geodesics) and then use some inductive procedure.

The present work can be considered as the anlogue of Nielsen's paper in the more difficult case where no simple axis exists. By a careful geometric analysis of the situation, again an extension of E from S^1 to D^2 is constructed thus giving the first completely constructive solution of the Nielsen realization problem (the first complete solution has been given by Kerckhoff in 1983 using methods of Thurston from Teichmüller theory). However, the present paper is formulated in a more general setting which has other important applications. In fact, one starts with a "convergence group" E acting on the circle S^1; this generalizes the action of a Fuchsian group on $S^1 = \partial D^2$ maintaining some of its essential features: for any sequence of distinct elements of E there exist $x, y \in S^1$ and a subsequence f_i such that on $S^1 - \{x, y\}$, $f_i \to y$ and $f_i^{-1} \to x$ uniformly on compact sets. The main result of the paper states that a convergence group is conjugate in Homeo(S^1) to the restriction of a Fuchsian group (this result has been obtained also by Casson and Jungreis using different 3- dimensional methods). For example, a group E acting on S^1 and containing the restriction of a Fuchsian group as a subgroup of finite index is a convergence group, thus the solution of the Nielsen realization problem. The other major application is the solution of the Seifert fiber space conjecture: if M is a compact orientable irreducible 3-manifold with infinite fundamental group containing an infinite cyclic normal subgroup $\mathbf{Z}$ then M is a Seifert fiber space (this is part of Thurston's geometrization conjecture for 3-manifolds). The solution depends on a result of Mess who showed that the quotient $\pi_1 M/\mathbf{Z}$ is (basically) an S^1-convergence group. By the main result of the present paper, the quotient is isomorphic to a Fuchsian group, therefore $\pi_1 M$ is isomorphic to the fundamental group of a Seifert fiber space and by a result of Scott M is homeomorphic to a Seifert fiber space.

This is certainly an impressive work bringing finally to a conclusion some old work of Nielsen.

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MSC:

57M50	General geometric structures on low-dimensional manifolds
20F65	Geometric group theory
57N10	Topology of general 3-manifolds (MSC2010)
30F99	Riemann surfaces
57N05	Topology of the Euclidean 2-space, 2-manifolds (MSC2010)
57R50	Differential topological aspects of diffeomorphisms

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simple axis; Seifert fiber space conjecture; irreducible 3-manifold with infinite fundamental group; infinite cyclic normal subgroup

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