Treves, F.

**Hypo-analytic structures: local theory.** (English) Zbl 0787.35003

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This monograph is devoted to study the local theory of the so-called hypoanalytic structures, a notion introduced by the author more than ten years ago, and which led subsequently to many new and interesting results, due to the author and his collaborators, M. S. Baouendi, C. H. Chang, P. D. Cordaro, N. Hanges, Jacobowitz, Rothschild.

The point of view is the following: take a real smooth manifold $M$, an overdetermined system of linear PDE defined by complex, smooth vector fields, under mild and natural assumptions (that these vector fields be the section of a vector subbundle $V$ of the complexified tangent bundle of $M$, and that this system be involutive). If more is supposed, i.e. that the structure thus defined be locally integrable, then, given a complete set of first integrals $Z_1, \dots, Z_m$ (where $m = \dim M - n$) of the system $(*)$ $L_j h = 0$, $j = 1, \dots, n$ where differentials are $\mathbb{C}$-linearly independent, it is easy to remark that any classical solution $h$ of $(*)$ is in fact the pullback of a CR function $\tilde{h}$ on the image of $Z$ (here CR means Cauchy-Riemann), i.e. $\tilde{h}$ is a solution of the $\overline{\partial}_b$-system on the image of $Z$. This leads to a new way of studying regularity conditions only by means of extendability properties of the corresponding $\tilde{h}$, and raises many new intriguing and interesting problems. All these can be best understood in the frame of what are called by the author hypoanalytic structures, i.e. an atlas of charts $(U, Z)$ where domains cover $M$, and the mappings $Z : U \to \mathbb{C}^m$ (supposed $C^\infty$) agree on overlaps up to biholomorhpisms. Obviously, complex analytic structures and CR structures are hypoanalytic.

The book starts by introducing the basic concepts and results. Chapter 1 is devoted to this, with some significant examples. Thus successively the notions of characteristic set of a formally integrable structure, of elliptic structure, of strongly noncharacteristic, totally real, maximally real submanifolds of a complex manifold are introduced, and the relations between them are studied. Interesting are the local representations in locally integrable structures. Much care is given to the Levi form in this context. The Frobenius theorems (in real and complex cases) are proved. Finally, involutive structures of finite type are introduced.

Chapter II, “Local approximation and representation on locally integrable structures” contains two essential results: the approximation formula (and its generalization) and the approximate Poincaré lemma. The approximation formula [*M. S. Baouendi* and *F. Treves*, Ann. Math., II. Ser. 113, 387-421 (1981; Zbl 0491.35036)] has already become a classical result and is often used. As its proof is a modification of Weierstrass’s initial proof of his classical polynomial approximation theorem, it is really surprising that such a result has been discovered so late. One also finds in this chapter some results on unique continuation of solutions.

At last in chapter III, “Hypoanalytic structures, hypocomplex manifolds” the notion of hypoanalytic structure is introduced, as well as that of hypoanalytic function and of hypoanalytic manifold, and their properties are studied. A paragraph is dedicatd to two-dimensional hypocomplex manifolds and another to the unique continuation of solutions in a hypoanalytic manifold.

Chapter IV, “Integrable formal structures. Normal forms” looks for invariants deduced from the Taylor expansions at a point of the coefficients of the vector fields $L_j$ $(j = 1, \dots, n)$ that span the tangent structure bundle $V$. The Hörmander numbers, multiplicities, weights, normal forms are introduced and studied in the formal case. All these are generalizations of results of *T. Bloom* and *I. Graham* [Invent. Math. 40, 217-243 (1977; Zbl 0346.32013)].

In fact, the main point is here the study of the so-called normal forms. Chapter V, “Involutive structures with boundary” carries over in this context what was done before. Here a natural dichotomy appears, between those structures in which the boundary is noncharacteristic, or is totally characteristic. The chapter ends with the study of the case of open connected subsets of $\mathbb{C}^n$, with $\mathcal{C}^\infty$ boundary with induced involutive structure. As an example of the result one can obtain, let us mention the following: any point $x_0 \in \overline{M}$ has an open neighborhood $U$ such that every holomorphic function $h$ on $M$ that grows slowly at the boundary can be approximated in $\mathcal{H}_{\text{slow}}(U)$ by a sequence of holomorphic polynomials.

Chapter VI (Local integrability and local solvability in elliptic structures) and VII (Examples of nonintegrability and nonsolvability), complement each other. In Chapter VI the author discusses the basic classes of structures in which both local solvability and local integrability hold. Local solvability is understood as local exactness, at some (or any) level, in the differential complex associated with the involutive structure. The prototype of a locally integrable structure in which exactness occurs at every level is the structure defined on $\mathbb{C}^{\nu}_{(z)} \times \mathbb{R}^{r}_{(t)}$ by the vector fields $\partial/\partial\bar{z}_1, \dots, \partial/\partial\bar{z}_\nu$, $\partial/\partial t_1, \dots, \partial/\partial t_{n-\nu}$.

In fact this structure is the local model of all CR structures when $\nu = 0$ and of all elliptic structures when $n - \nu = r$. For the vanishing of the local cohomology of the associated differential complex it suffices to have homotopy formulae, and it suffices to establish these formulae in the elliptic case. The author has chosen to follow the Bochner-Martinelli-Koppelman-Leray method (in convex domains), with Hölder-norm estimates of the corresponding homotopy operators, in the complex case. These, combined with the Newton method, leads to the proof of the Newlander-Nirenberg theorem [*S. M. Webster*, Math. Z. 201, No. 3, 303-316 (1989; Zbl 0668.32005)]. After that, using the Newlander- Nirenberg theorem, one shows that the elliptic structures are locally integrable, which shows that they can be modelled after $\mathbb{C}^{\nu} \times \mathbb{R}^{n-\nu}$. Once this is established one can obtain homotopy formulae in $\mathbb{C}^{\nu} \times \mathbb{R}^{n-\nu}$. Elliptic structures are a subclass of what is called Levi flat structures, which are known to be locally integrable. The author proves here a weaker result. The last two paragraphs are concerned with involutive structures locally invariant under a transverse group action, and in this case, the base manifold can be locally modelled after the product of a Lie group with a manifold for which the induced structure is an elliptic one.

As said, chapter VII, consists of examples of involutive structures on which local solvability does not hold, or which are not locally integrable. The first example is the so-called Mizohata structure (i.e. such that $V$ has rank one, the characteristic set $T^0 \neq 0$ and the Levi form is nondegenerate at every point of $T^0\backslash 0$). It is shown that in this case, if the signature of the Levi form is $|n-2|$ (supposing $n > 1$, and $m = 1$), we have nonsolvability and non- integrability. More precisely, in a neighborhood of 0 one can perturb a locally integrable structure on $M$ to give a new involutive structure which is non-integrable (still involutive) and in such a way that it is a Mizohata structure.

Let us also mention the study of Mizohata structures on two-dimensional manifolds, and the case when the cotangent structure bundle has rank one.

Next, the author introduces the Lewy structure (a CR structure, with the characteristic set $T^0_a$ a line bundle, with nondegenerate Levi form on $T^0\backslash 0$), and studies the non-integrability in these structures.

Chapter VII "Necessary conditions for vanishing of the cohomology. Local solvability of a single vector field", begins with establishing necessary conditions for exactness. In particular, such a condition based on the Levi form is given. Let us mention the following result: If the equation $d'u = f$ is locally solvable at the point 0 in degree $q$, then every germ at 0 of a hypoanalytic function whose differential spans the characteristic set at 0 is acyclic in dimension $q$. (Here $d'$ is the differential of the complex associated to the structure subbundle $V$). As an application, one gets that if $m = 1$, and $d'u = f$ is locally solvable at 0 in degree $q$, the locally integrable structure of $M$ is acyclic at 0 in dimension $q$. There are also applications when $m > 1$.

The chapter ends with the analysis of the case of a single vector field and the study of the so-called property $(\mathcal{P})$; in particular condition $(\mathcal{P})$ implies the existence of $L^2$-solutions [the author, Am. J. Math. 92, 369-380 (1970; Zbl 0236.35040)]; its necessity is essentially an argument of Moyer [see the author, Am. J. Math. 112, No. 3, 403-421 (1990; Zbl 0763.35022)].

Chapter IX, "FBI transform in an hypoanalytic manifold" has a special flavour, and makes the reader eager waiting for the second volume of this treatise. Because here for the first time the microlocal aspects of the problems explicitly appear and it seems that this is really the heart of the matter. In this chapter we have to content ourselves with the simplest local definition, and to see how effective the FBI transform is, by the proof of the propagation of hypoanalyticity along elliptic submanifolds.

The last chapter, "Involutive systems of nonlinear first-order differential equations" gives, via a microlocal approach, the beginning of an answer to the question of how much of the theory of locally integrable structures can be carried over to systems of first order nonlinear partial differential equations. The method used here, by the Hamiltonian lift to the (complexified) 1-set bundle over the base manifold is in fact an adaptation of the classical method of characteristic. The involutiveness is defined by means of holomorphic Poisson brackets, and the result on the uniqueness in the Cauchy problem follows closely *G. Métivier* [Invent. Math. 82, 263-282 (1985; Zbl 0594.35018)] (the semilinear case was first attacked by *M. S.*

*Baouendi*, *C. Goulaouic* and the author [Commun. Pure Appl. Math. 38, 109-123 (1985; Zbl 0609.35019)]).

The book has a very personal character; it is far from being encyclopedic. For instance, there are results in the analytic case (see for instance Trepreau's Conference at the Bourbaki Seminar) that are not even mentioned. But a correct appreciation can be made only after the publication of the second volume, which we await as soon as possible.

As it is, it is a *must* for those people interested in PDE and (or) Complex Analysis (specially CR-specialists) and it has the advantage of being practically self-contained and easy to read, again the unmistakable mark of the author's personality. In fact, F. Treves has opened a new way with important results, and this first volume is an introduction to this.

Reviewer: G.Gussi (Bucureşti)

**MSC:**

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| 35-02 | Research exposition (monographs, survey articles) pertaining to partial differential equations |
| 35F05 | Linear first-order PDEs |
| 32V05 | CR structures, CR operators, and generalizations |
| 35N99 | Overdetermined problems for partial differential equations and systems of partial differential equations |
| 58J10 | Differential complexes |
| 35A20 | Analyticity in context of PDEs |
| 35F20 | Nonlinear first-order PDEs |
| 32W05 | $\overline{\partial}$ and $\overline{\partial}$-Neumann operators |
| 35N15 | $\overline{\partial}$-Neumann problems and formal complexes in context of PDEs |
| 32-02 | Research exposition (monographs, survey articles) pertaining to several complex variables and analytic spaces |
| 58J60 | Relations of PDEs with special manifold structures (Riemannian, Finsler, etc.) |



**Keywords:**

FBI transform; hypoanalytic structures; CR function; local representations; locally integrable structures; Levi form; approximation formula; approximate Poincaré lemma; hypocomplex manifolds; hypoanalytic function; hypoanalytic manifold; local solvability; local integrability; differential complex; local cohomology; homotopy formulas; elliptic structures; involutive structures; Mizohata structure; Lewy structure; systems of first order nonlinear partial differential equations